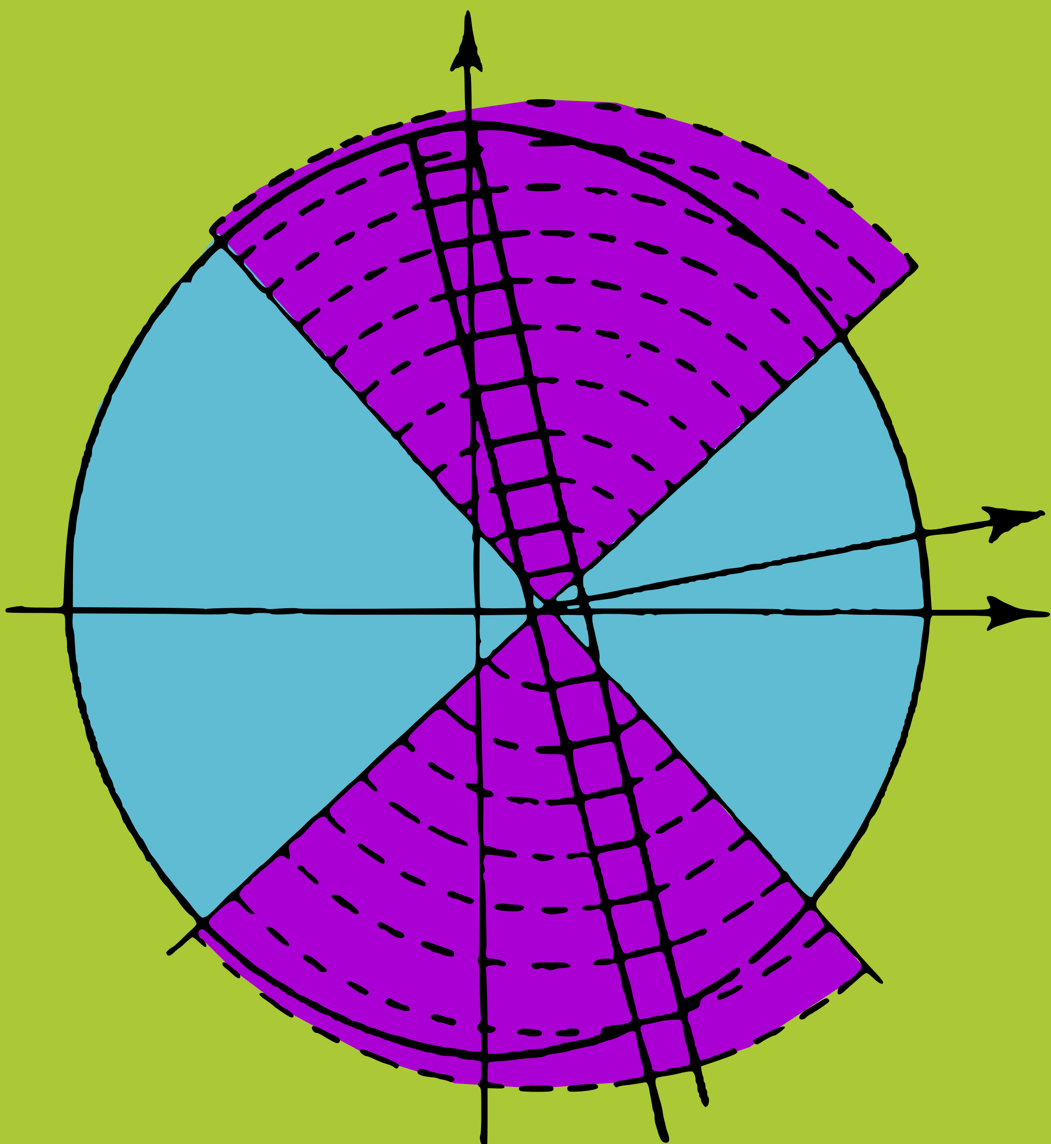
B.B. Kadomtsev

PLASMA TURBULENCE



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By

B. B. KADOMTSEV

Translated from the Russian by

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PREFACE

THIS book is a translation of a review by Academician B. B. Kadomtsev of the I. V. Kurchatov Institute, Moscow, which was written at my suggestion in order to have available in concise form the essence of the large body of papers on this difficult yet important subject. It was originally published in Russian in Volume 4 of "Problems in Plasma Theory", edited by M. A. Leontovich (1964).

Academician Kadomtsev is a leading authority on plasma turbulence, and he and his colleagues have contributed many of the original papers on the subject. It is now well known that some form of turbulence is very frequently and often disastrously present in experiments on the confinement of hot plasma. It is not impossible that in the quest for a thermonuclear plasma we shall ultimately have to deal with a more or less turbulent state, and the understanding of this state is therefore important to nuclear fusion research. I believe this book will contribute greatly towards such an understanding and help to stimulate further work on the subject.

The translation and editing of the book have been carried out by Mr. L. C. Ronson and Dr. M. G. Rusbridge of the Culham Laboratory. I am grateful to them and to Mrs. Mary Hardaker who typed the manuscript.

J. B. ADAMS

April 1965

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CONTENTS

PREFACE v

NOTATION viii

INTRODUCTION 1

I. A QUASI-LINEAR APPROXIMATION 5

1. Instability and Turbulence 5

2. Laminar Convection of a Plasma 7

3. Quasi-linear Approximation in Kinetics . . . 15

II. INTERACTION OF WAVES AT WEAK TURBULENCE . . 31

1. Kinetic Wave Equation 31

2. Interaction of the Waves in the Plasma with consideration of the Thermal Motion of the Particles . . . 41

III. METHODS OF CONSIDERING STRONG TURBULENCE . 49

1. The Weak Coupling Approximation 49

2. Phenomenological Approach to a Description of a Strong Turbulence 58

IV. TURBULENCE IN A PLASMA 65

1. Non-linear Damping of Langmuir Waves 65

2. Excitation of Ion Oscillations by an Electron Current . 68

3. Drift Instability of a Plasma 78

4. Turbulent Diffusion of a Plasma 106

5. Turbulent Plasma in Experimental Conditions . 123

CONCLUSION 139

REFERENCES . . . 141

INDEX 147

Notation

THE notation conforms as closely as possible to that of the original, which is in general conventional or explained in the text. The only major exception is the substitution of curl for rot, which is not so familiar to English readers. Vectors are shown by bold type. The vector product is denoted by square brackets, thus [**hk**]. No special notation is used in general for the scalar product, but round brackets, thus (**kk**′), may be used if required for clarity. Otherwise two adjacent vectors imply the scalar product. Note the appearance of the scalar triple product in equations such as (IV. 128) and (IV. 129).

INTRODUCTION

It is well known that a real plasma is rarely quiescent; as a rule many forms of noise and oscillation arise spontaneously in the plasma. Langmuir pointed out that these fluctuations represent more than just harmless oscillations about an equilibrium position and often wholly determine the character of the phenomena occurring in the plasma. In particular, interaction between electrons and oscillations accounts for the " strong " scattering of electrons in a gas discharge, first described by Langmuir (1, 2). The collective interaction of particles also accounts for the well-known Langmuir paradox; namely, that even at very low gas pressure the velocity distribution of the electrons in a glow discharge is, to a high degree of accuracy, a Maxwellian distribution. The decisive part played by oscillations in the interaction of an electron beam with a plasma has been demonstrated in the papers of Merrill and Webb (3) and Looney and Brown (4).

Experiments with plasmas in a magnetic field and in particular experiments on magnetic containment of a high temperature plasma in connection with controlled thermonuclear reactions have revealed further unexpected phenomena essentially connected with oscillations in the plasma. Prominent amongst these is the " anomalous " diffusion of a plasma across a magnetic field. This effect was first observed by Bohm, Burhop and others while investigating the operation of ion sources (5) and later the enhanced diffusion of a plasma, related to its instability, was observed in a series of experimental devices.

Following the work of Bohm (5), who suggested that the enhanced diffusion of a plasma is due to random oscillations of the electric field set up by an instability, the term " turbulence " has been increasingly applied to this process. At present we understand by turbulence the motion of a plasma in which a large number of collective degrees of freedom are excited. Thus, when applying the term " turbulence " to a plasma, it is used in a broader sense than in conventional hydrodynamics. If hydrodynamic turbulence represents a system made up of a large number of mutually interacting eddies, then in a plasma we have together with the eddies (or instead of them), also the possible excitation of a great variety of oscillations. Depending on the degree of freedom which is excited, the character of the interaction between the excitations may vary considerably.

During the eddy motion of an ordinary fluid the separate eddies, in the absence of their mutual interaction, do not propagate in space. When their interaction is included the eddies " spread out " in space with time, though the corresponding velocity is not large and therefore each separate eddy has a

considerable time available to interact with its neighbours. In this case we are faced with a strong interaction of excitations and correspondingly with a strong turbulence. On the other hand, during a wave motion the separate wave packets can separate from one another over large distances. In this case the interaction of separate wave packets with one another is weak, and we can therefore refer to a weak turbulence. The motion of the plasma in the weakly turbulent state, constituting a system of weakly correlated waves, shows greater similarity to the motion of the wavy surface of the sea or the oscillations of a crystal lattice than to the turbulent motion of an ordinary fluid.

The theoretical consideration of a weakly turbulent state is considerably facilitated by the possibility of applying perturbation theory, i.e. an expansion in terms of a small parameter such as the ratio between the energy of interaction between the waves and their total energy. The problem of the non-linear interaction between waves in a plasma has been considered by Sturrock (6) for the special case of interaction between Langmuir waves in a cold homogeneous plasma.

For the case of very small amplitude, when the interaction between the oscillations can be neglected, one can use the so-called quasi-linear approximation in which only the reaction of the oscillations on the average velocity distribution function of the particles is considered. The quasi-linear approximation was referred to by Romanov and Filippov (7) and has been further developed in papers by Vedenov *et al.* (8), as well as by Drummond and Pines (9). Section I of the present review is devoted to the quasi-linear approximation.

Unfortunately the quasi-linear method has only a fairly narrow field of application, since non-linear interaction of the oscillations already begins to play a considerable part at not very large amplitudes. In the paper by Camac *et al.* (10), the non-linear interaction of Alfvén- and magneto-sonic waves is described by the kinetic wave equation, which is well known in solid state theory (11). Camac *et al.* (10) applied this method to describe the structure of a collisionless shock wave. The problem has been considered in somewhat greater detail by Galeev and Karpman (13), while the interaction of Langmuir waves has been studied in reference (19).

In the simplest variant of the kinetic wave equation only three-wave processes are considered, namely the decay of the wave $\mathbf{k}$, ω into two waves $\mathbf{k}'$, ω' and $\mathbf{k}''$, ω'', and the merging of two waves into one. Such processes are important only for dispersion relations $\omega_{\mathbf{k}} = \omega(\mathbf{k})$ for which it is possible to satisfy simultaneously the laws of conservation of energy and of momentum: $\mathbf{k}'' = \mathbf{k}-\mathbf{k}'$, $\omega_{\mathbf{k}''} = \omega_{\mathbf{k}}-\omega_{\mathbf{k}'}$. When these conditions are not satisfied, scattering of the waves by the particles is a more important process and can be taken into account only on the basis of a full kinetic theory. Such a theory has been published by Kadomtsev and Petviashvili (14), who obtained a kinetic wave equation including the thermal motion of the particles. This equation was obtained later by a slightly different method by Karpman (15). The kinetic wave equation is deduced and considered in Section II.

Unfortunately, in numerous practical cases one is faced not by weak but by strong turbulence. In particular, strong turbulence is related to an anomalous diffusion of the plasma across the magnetic field. To determine the fluctuation spectrum in a strongly turbulent plasma and the effect of these fluctuations on the averaged quantities, it is sometimes possible to use the analogy with ordinary hydrodynamics and, in particular, to apply a phenomenological description of the turbulent motion.

Such an approach has been used by this author in two specific problems, referring to turbulent diffusion of a plasma in a trap with magnetic mirrors (16) and in the positive column of a glow discharge (17). In these cases the concept of the mixing length was used. The results of this discussion are in good agreement with experimental data, which indicates the success of this approach.

However, in a plasma other strongly turbulent motions which are different from the eddy motion of an ordinary fluid may develop. It is therefore desirable to have available more systematic methods for describing strong turbulence. In our view, such a method may be the weak coupling approximation discussed in Chapter III. In this approximation, which ought preferably to be called the intermediate coupling approximation, the turbulent motion is described by a system of non-linear integral equations for the spectral density $I_{\mathbf{k}\omega}$ and the Green's function $G_{\mathbf{k}\omega}$ describing the response of the system to an external force. As the coupling between the oscillations decreases, this system of equations goes over into the kinetic wave equation.

In conventional hydrodynamics, the weak coupling equations have been obtained by Kraichnan (18) who showed that in their simplest form the weak coupling equations lead to a spectrum which is different from Kolmogorov's spectrum in the region of large k. As will be shown in Section III.2, the reason is that in Kraichnan's equations the adiabatic character of the interaction of the short wave with the long wave pulsations is not taken into account. The consideration of this adiabatic interaction makes it possible to obtain improved weak coupling equations.

In Section IV, specific examples of turbulent processes in a plasma are considered. In particular the interaction of Langmuir waves and the excitation of ion oscillations by an electron current are considered, but the main attention is devoted to the turbulent diffusion of a plasma in a magnetic field.

As we have mentioned earlier, the turbulent diffusion problem goes back to Bohm (5), who put forward the hypothesis that an inhomogeneous plasma in a magnetic field must always be unstable because of the presence of a drift current of the electrons relative to the ions. If this be in fact so, the corresponding instability must lead to a turbulent ejection of the plasma with a velocity of the order of the drift velocity. According to Bohm, this process can be considered phenomenologically as a diffusion with coefficient of diffusion of the order

$$D_{\mathrm{B}} = \frac{10^4 T}{H}$$

where T is the electron temperature in electron volts and H the magnetic field in kilogauss.

Bohm's argument gave rise to the illusion of a universal validity for this coefficient and as a result attempts to obtain Bohm's coefficient from more general considerations have continued to this day. It has now become evident, however, that the coefficient of turbulent diffusion cannot be obtained without a detailed investigation of the instability of an inhomogeneous plasma and in particular of its drift instability.

Investigations of the drift instability of a plasma in a magnetic field were started by Tserkovnikov (19) who limited his investigations to perturbations constant along the direction of the magnetic field. Rudakov and Sagdeev (20) went a step further by considering instabilities at oblique angles with a transverse wavelength considerably larger than the mean Larmor radius of the ions. The complete investigation of the drift instability has been carried out only quite recently, following the work of Rosenbluth, Krall and Rostoker (21), with the investigation of perturbations with transverse wavelength of the same order as the mean ion Larmor radius. The principal results in this field were obtained by Mikhailovskiĭ (22). In a dense plasma where an important part is played by collisions between the particles, the drift instability changes to a drift-dissipative instability first observed by Timofeev (23). In a note by Sagdeev and Moiseev (24), it was stated that this drift-dissipative instability may lead to an escape of plasma of the order of the Bohm diffusion.

The principal results of the investigations of the drift instability of a plasma are discussed in Section IV.3. The following section considers the problem of the type of diffusion resulting from the drift instability in specific conditions.

In Section IV.5, various turbulent processes in specific experimental devices are described and briefly discussed. Turbulent processes are considered in toroidal discharges and in magnetic traps and brief reference is also made to experimental data on turbulent heating and diffusion of plasmas.

In writing this review, we have aimed at maximum clarity of expression; where rigour appears to conflict with simplicity, simplicity is given preference. Attention has been given chiefly to processes occurring under laboratory conditions; problems of astrophysical application are completely omitted. In particular no reference is made in the present review to a broad group of investigations in which the turbulence of an ideal conducting fluid (i.e. magnetohydrodynamic turbulence) and turbulent shock waves are considered.

I. A QUASI-LINEAR APPROXIMATION

1. Instability and Turbulence

As a rule, turbulence develops as a result of an instability of an initial laminar state. In order to visualise the transition from the laminar to the turbulent state, it is convenient to examine the behaviour of the system while changing some parameter, R, the increase of which results in a loss of stability. In conventional hydrodynamics such a parameter is the Reynolds number, whilst in a plasma there are a number of cases where the magnetic field strength plays a similar role.

Let us recall first what occurs during the loss of stability of a system with one degree of freedom, for instance, a valve oscillator, where the part of the parameter R is played by the feed-back. If the feed-back R is smaller than some critical value R_c then all small oscillations are damped. When $R > R_c$, on the other hand, the state with zero amplitude is unstable and oscillations of finite amplitude are excited. In other words, the value $R = R_c$ is a bifurcation point above which the oscillator is in an excited state.

In these conditions two types of excitations are possible—a soft type and a hard type. In the soft regime the amplitude of the oscillation varies continuously with R, vanishing for $R = R_c$ (see Fig. 1). In the hard regime the amplitude increases abruptly to some finite value as soon as the value R exceeds the value R_c and when R decreases it drops abruptly to zero for $R = R_0 < R_c$ (see Fig. 2). In the region $R_0 < R < R_c$ the circuit is stable with respect to infinitely small perturbations, but unstable to perturbations of sufficiently large amplitude. The unstable equilibrium state is represented in Fig. 2 by the

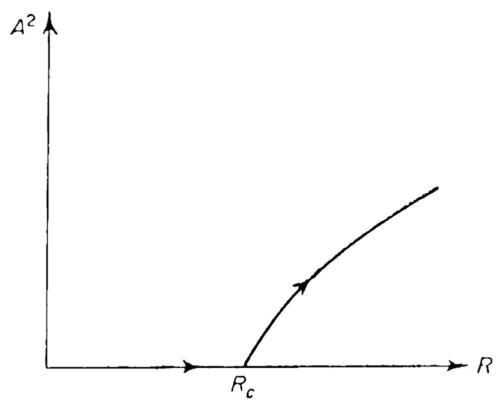

FIG. 1. Soft excitation

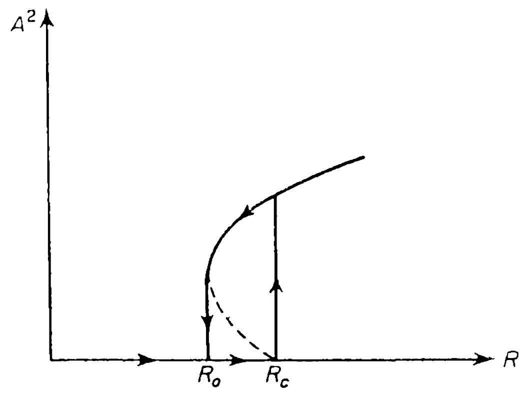

FIG. 2. Hard excitation

dashed line. In the case of a small non-linearity of the system this picture of the excitation is fully described by means of the widely known Van de Pol method, i.e. a small amplitude expansion.

Landau (25) has shown that the transition to the turbulent state also constitutes an excitation of the system as a result of an instability. To examine Landau's argument let us consider a continuous medium, i.e. a system with an infinite number of degrees of freedom, and assume that the excitation is soft. Then, as R is increased the following picture will be observed. For small R we have the laminar state in which all quantities are completely defined by the initial and boundary conditions, i.e. the system has no superfluous degrees of freedom. For R greater than some critical value R_1 a normal mode of the system is excited whose amplitude increases monotonically with R; in other words, an additional degree of freedom appears in the system. As R is further increased further degrees of freedom may be excited and ultimately we arrive at a turbulent state.

When R only slightly exceeds R_1, the amplitude A can be determined by expanding with respect to $R-R_1$, similarly to Van de Pol's method. This method was first applied to the problem of thermal convection of a fluid

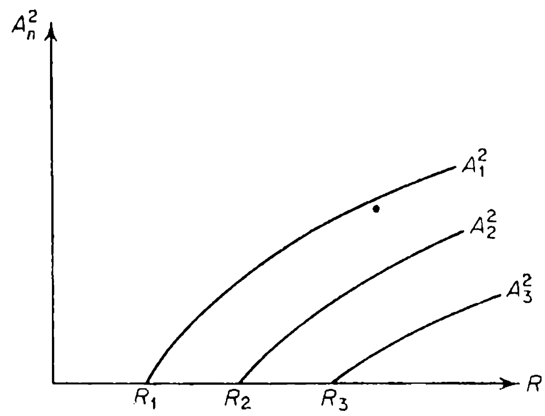

FIG. 3. Mild excitation of a system with many degrees of freedom

(Sorokin (26)) and was later used by Stuart (27) to describe the eddy cells in a fluid between rotating cylinders. This method is now known as the quasi-linear method and leads to satisfactory results for $R-R_1 \ll R_1$.

The behaviour of the system as R is increased also depends on its specific properties. We may find that numerous strongly interacting modes are excited and in this case the transition to turbulence takes place fairly rapidly. Alternatively we may find that all the higher modes are harmonics of the fundamental mode first excited and are synchronised with it. In other words, we have a non-linear oscillation of finite amplitude which really represents only one degree of freedom. (The losses in the positive column of a glow discharge are an example of this situation.) Finally, there is the possibility that the excited modes will interact only weakly with one another and a weakly turbulent motion will develop. If the interaction between the modes can be neglected, weak turbulence can be described by the quasi-linear approximation.

2. Laminar Convection of a Plasma

Let us consider two simple examples where the instability of a plasma leads to the appearance of a convective flow.

(a) *Convection of a Weakly Ionized Plasma in an Inhomogeneous Magnetic Field*

As a first example, let us consider the convection of a weakly ionized plasma in an azimuthal (toroidal) magnetic field (28). As is well known, a fully ionized plasma in such a field is convectively unstable; because of its diamagnetism it is pushed out radially. If the magnetic field is sufficiently high, then a weakly ionized plasma is subject to a similar instability. Let us consider here what happens to a plasma when the magnetic field exceeds the critical value H_c at which the instability first appears.

Suppose that the plasma is located between two ideally conducting cylinders of radius R and $R+d$ respectively, ($d \ll R$), and ionization is achieved in such a way (for instance using heated grids in caesium vapour) that a constant density n is maintained at the inner cylinder and a density $n-\delta n$ at the outer cylinder, where $\delta n \ll n$.

We shall assume both the electrons and the ions are " magnetized ", i.e. $\Omega_j \tau_j = \dfrac{eH}{m_j c}\tau_j \gg 1$, where τ_j is the mean free time and m_j the mass of particles of type j. Then from the equation of motion for the electrons

$$T_e \nabla n = en\nabla\varphi - \frac{en}{c}[\mathbf{v}_e \mathbf{H}] - \frac{m_e n}{\tau_e}\mathbf{v}_e \tag{I. 1}$$

we obtain the following

$$n\mathbf{v}_e = \frac{c}{H}\left[\mathbf{h}, n\nabla\varphi + \frac{T_e}{e}\nabla n\right] - D_\perp \nabla n + n b_\perp \nabla\varphi \tag{I. 2}$$

where $D_{\perp} = \dfrac{D_e}{\Omega_e^2 \tau_e^2} = \dfrac{T_e}{m_e \tau_e \Omega_e^2}$ is the transverse diffusion coefficient, φ the potential of the electric field, T_e the electron temperature and $b_{\perp} = \dfrac{eD_{\perp}}{T_e}$ the transverse mobility of the electrons. Since we assume $\Omega_e \tau_e \gg 1$, the last term in I.2 which contains $b_{\perp}$ can be neglected.

Substituting I.2 into the continuity equation and neglecting the curvature of the magnetic field, we obtain

$$\frac{\partial n}{\partial t} + \mathbf{v}\nabla n = D_{\perp}\Delta n \tag{I. 3}$$

where $\mathbf{v} = \dfrac{c}{H}(\mathbf{h}\nabla\varphi)$ is the drift velocity in crossed electric fields and $\mathbf{h} = \mathbf{H}/H$. On the other hand, from the ion equation of motion, assuming $T_i = 0$, we have

$$\mathbf{v}_i = \frac{c}{H}[\mathbf{h}\nabla\varphi] - \frac{e\tau_i}{m_i(\Omega_i \tau_i)^2}\nabla\varphi \tag{I. 4}$$

We now substitute the expressions for $n\mathbf{v}_e$ and $n\mathbf{v}_i$ in the quasi-neutrality condition div $n(\mathbf{v}_i - \mathbf{v}_e) = 0$. Considering that

$$\operatorname{div}\frac{[\mathbf{h}\nabla n]}{H} = \frac{\partial n}{\partial z}\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{H}\right) \cong \frac{2}{RH}\,\frac{\partial n}{\partial z},$$

we obtain

$$\frac{c}{H}\operatorname{div}(n\nabla\varphi) = -\frac{2D_a}{R}\frac{\partial n}{\partial z} \tag{I. 5}$$

where $D_a = \dfrac{T_e \tau_i}{m_i}$ is the ambipolar diffusion coefficient in the absence of a magnetic field.

In the equilibrium state $n = n(r)$, $\varphi = \varphi(r)$, and consequently there are no convective fluxes. Let us consider the value of the critical magnetic field H_c at which convection starts. Since $\delta n \ll n$, the density n can be assumed constant on the left hand side of relation I.5 and the perturbation of the density and potential can be chosen in the form $\sin\dfrac{\pi x}{d}\exp(ikz - i\omega t)$ where $x = r - R$. The linearised eqns. I.3 and I.5 then take the form

$$[-i\omega + D_{\perp}(k^2 + k_0^2)]n^{(1)} = -ik\frac{c\varphi^{(1)}}{H}\frac{dn}{dx} \tag{I. 6}$$

$$\frac{c}{H}(k^2 + k_0^2)\varphi^{(1)} = -ik\frac{2T_e\tau_i}{m_i R n}n^{(1)} \tag{I. 7}$$

where $k_0 \equiv \pi/d$. From this we obtain

$$-i\omega = -D_{\perp}(k^2 + k_0^2) + \frac{2D_a}{R}\frac{k^2}{k^2 + k_0^2}\frac{\delta n}{nd}$$

For $\omega = 0$, i.e. $D_\perp = \frac{2D_a}{R}\frac{k^2}{(k^2+k_0^2)^2}\frac{\delta n}{nd}$ the plasma becomes unstable. The maximum value of $D_\perp$ is attained for $k = k_0$ and is equal to

$$D_c = \frac{D_a d}{2\pi^2 R}\frac{\delta n}{n} \tag{I. 8}$$

Clearly $D_c \ll D_a$. The critical field H_c is defined by the relation $D = D_c$; for $H > H_c$, i.e. $D < D_c$, laminar convection develops in the plasma.

When H is only slightly larger than H_c the convective flow can be determined using the quasi-linear method. The total diffusion flux is defined by the relation $q = q_0 + \delta q$, where $q_0 = D_\perp \frac{dn}{dx}$, $\delta q = \langle n^{(1)} v_x^{(1)} \rangle$ and the angular brackets indicate averaging with respect to time. In the first approximation $v_x^{(1)}$ can be expressed in terms of $n^{(1)}$ with the aid of (I.7) and for $n^{(1)}$ we insert the dependence on x and z given above. Integrating the resulting expression for the flux with respect to x with $q = \text{const}$ gives

$$D_\perp \delta n + \frac{1}{2}\frac{D_a nd}{R}A^2 = qd \tag{I. 9}$$

where A is the amplitude of the density oscillations defined by

$$\frac{n^{(1)}}{n} = A \sin\frac{\pi x}{d} e^{ikz - i\omega t}$$

A second relation connecting q and A^2 is obtained by substituting $\varphi^{(1)}$ obtained from (I.7) into (I.6), setting $\omega = 0$, and multiplying the result by $n^{(1)}$. Representing the derivative of the density in the form

$$\frac{dn}{dx} = -\frac{q}{D_\perp} + \frac{\langle n^{(1)} v_x^{(1)} \rangle}{D_\perp}$$

averaging the result with respect to x, and using (I.8), we obtain

$$\frac{D_\perp^2}{D_c}\delta n + \frac{3}{4}\frac{D_a nd}{R}A^2 = qd \tag{I. 10}$$

From the relations (I.9) and (I.10) we obtain

$$A^2 = \left(\frac{\delta n}{n}\right)^2 \frac{2}{\pi^2}\left(1 - \frac{D_\perp}{D_c}\right) \tag{I. 11}$$

$$q = q_0 + \frac{2\delta n D_\perp}{d}\left(1 - \frac{D_\perp}{D_c}\right) \tag{I. 12}$$

Thus, the flux q depends on the magnetic field as shown in Fig. 4. For $H > H_c$ the flux increases instead of decreasing with increasing field.

This picture is completely analogous to the picture of the development of convective cells in thermal convection between two parallel plates; here the onset of convection is also sharp. In more realistic conditions of a chamber

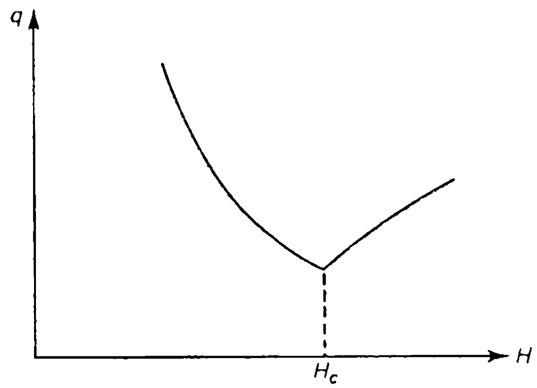

FIG. 4. Plasma flux against magnetic field during development of convection

bounded along the magnetic field, there may be no well-defined critical field, the flux q varying smoothly with the field.

The increase of the effective diffusive loss due to convection has been clearly observed by Bostick and Levine (29) and has been investigated more recently by Golant *et al.* (30), who investigated the decay of a plasma in a toroidal magnetic field. The plasma was contained in a toroidal glass vessel of major radius R with a circular cross section of radius a. A fairly long interval of time was allowed for the density distribution to become established and the decay of the plasma density was then observed and found to follow an exponential law $n = n_0 e^{-t/\tau}$.

The loss can easily be determined by assuming approximately that the convective velocity parallel to r can be neglected, i.e. $\dfrac{\partial \varphi}{\partial z} \cong 0$. We then obtain from (I.5) $\mathbf{v} = \dfrac{2D_a}{R}\,\mathbf{e}$, and eqn. (I.3) takes the form

$$\frac{\partial n}{\partial t} + \frac{2D_a}{R}\mathbf{e}\nabla n = D_\perp \Delta n \tag{I. 13}$$

where $\mathbf{e}$ is the unit vector parallel to the x axis.

Neglecting the field curvature in the expression for the Laplacian, eqn. (I.13) can be considered as the equation for diffusion in a straight cylindrical tube in the presence of a transverse flux. It is easily seen that the solution of eqn. (I.13), with the boundary condition that the density at the wall vanishes, is given by the expression

$$n = n_0 \exp\left(-\frac{t}{\tau} + \frac{v}{2D_\perp}x\right) J_0\left(\frac{\alpha_0 r}{a}\right) \tag{I. 14}$$

where x is now measured from the tube axis. In (I.14) n_0 = const, $\alpha_0 \equiv 2{\cdot}4$ is the first root of the zero order Bessel function J_0 and the decay constant is given by the relation

$$\frac{1}{\tau} = \frac{1}{2\tau_0}\left(\frac{D_\perp}{D_0} + \frac{D_0}{D_\perp}\right) \tag{I. 15}$$

where

$$\tau_0 = \frac{a}{\alpha_0 v}, \qquad D_0 = \frac{av}{2\alpha_0}, \qquad v = \frac{2D_a}{R}.$$

According to (I.15), the decay constant as a function of the magnetic field must pass through a minimum at $D_\perp = D_0$. In Fig. 5, which was

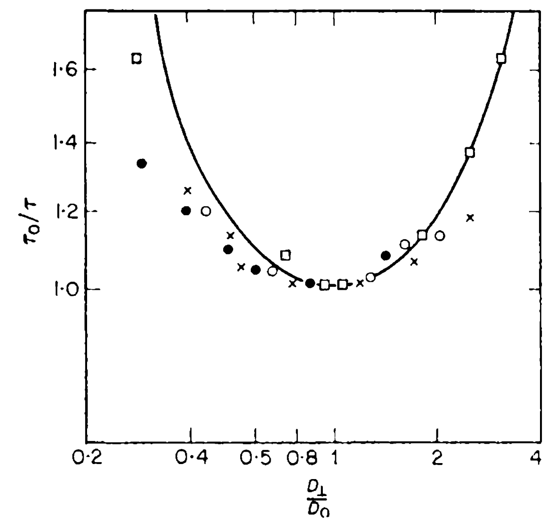

FIG. 5. Lifetime of a decaying plasma in a toroidal tube (helium, R = 28 cm, a = 0·3 cm)

● p = 0·025 mm Hg
○ p = 0·04 mm Hg
× p = 0·055 mm Hg
□ p = 0·12 mm Hg

taken from (29), a comparison is given of the experimentally determined relationship between τ_0/τ and $D_\perp/D_0$ (where $1/\tau_0$ and D_0 are the decay constant and the diffusion coefficient respectively at the minimum) with the theoretical formula (I.15). As may be seen, the experimental results are in good agreement with the theory.

(b) *Convection of the Plasma of the Positive Column in a Magnetic Field*

Let us now consider the convection of a plasma in a homogeneous magnetic field in the presence of a longitudinal current. The development of convection of a weakly ionized current-carrying plasma was described by Lehnert (31) while investigating the diffusion of charged particles from the positive column of a glow discharge located in a strong magnetic field. It has also been studied in greater detail in a paper by Lehnert and Hoh (32).

In these investigations the relationship between a longitudinal electric field E and the magnetic field H was studied. In the positive column a decrease of the diffusion coefficient leads to a decrease of the electric field, so that if the diffusion is classical, the electric field E should monotonically

decrease as the magnetic field is increased. This relationship between E and H is in fact observed experimentally, but only for not too strong magnetic fields. As soon as H exceeds a critical value H_c, the dependence of E on H changes markedly: the electric field begins to increase, ultimately attaining a saturation value E_s (Fig. 6).

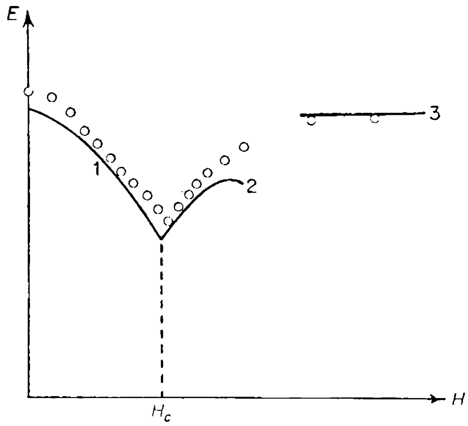

FIG. 6. Dependence of longitudinal electric field on magnetic field in the positive column of a helium discharge: 1. Region of classical diffusion. 2. Region of laminar convection (helical discharge). 3. Region of turbulence

The general nature of this effect was explained in a paper by the author and Nedospasov (32) where it was shown that the positive column of a glow discharge loses its stability for $H > H_c$. The mechanism of this instability, which is known as the current-convective instability, can be illustrated by considering a plasma filament slightly curved along a helical line. To fix ideas suppose the discharge is vertical, then in the presence of a longitudinal electric field E, which we suppose directed upwards, the electron " corkscrew " of the curved filament is displaced downwards relative to the ion corkscrew and charges appear at the surface of the filament, namely a positive charge at the upper surface and a negative one at the lower surface. These charges set up an azimuthal electric field $E_\perp$ which leads to a drift of the plasma along the radius, i.e. to an increase of the initial perturbation. In a weak magnetic field this effect is suppressed by the diffusion. In a strong magnetic field, when the transverse diffusion decreases to such an extent that it cannot eliminate the density perturbations, this mechanism leads to instability of the plasma filament with respect to a helical distortion.

To estimate the value of the critical field, let us consider the stability problem in the W.K.B. approximation; this is accurate for short wave perturbations only, but describes qualitatively correctly perturbations with a longer wavelength, of the order of the tube radius a. In this approximation it is assumed that the density of the plasma is a slowly varying function of a co-ordinate x, and the perturbation is selected in the form of a plane wave $\exp(-i\omega t + i\mathbf{kr})$. At a neutral gas pressure of the order of 1 mm Hg, the

instability begins when $\Omega_i\tau_i < 1$, so that the ion velocity can be assumed to be given by $\mathbf{v}_i = -b_i D\varphi$. Substituting this expression into the continuity equation we obtain for the perturbation of the ion density the following expression:

$$\frac{n_i}{n} = -i\frac{b_i k^2}{\omega}\varphi \tag{I. 16}$$

where φ is the perturbation of the electric field potential.

The perturbation of the electron density n_e can be determined from the continuity equation in which the electron velocity $\mathbf{v}_e$ is given by the relation (I.2). In the W.K.B. approximation the linearised continuity equation for the electrons takes the form

$$\left(-i\omega+k_z^2 D_e+k_\perp^2 D_\perp+ik_z u-ik_y\frac{cE_x}{H}\right)n_e-\left(\frac{ik_y c}{H}\frac{dn}{dx}+b_e k_z^2+b_\perp k_\perp^2\right)\varphi=0 \tag{I. 17}$$

where b_e, D_e are the longitudinal, and $b_\perp$, $D_\perp$ the transverse mobility and diffusion coefficient respectively of the electrons, $u = b_e E$ the mean velocity of the electrons in the longitudinal electric field E, and E_x the transverse electric field in the equilibrium state, which in the case of ambipolar diffusion is given by

$$E_x = -D_e b_e^{-1}(1+y)^{-1}\frac{d\ln n}{dx}$$

$\left(y = \frac{b_i}{b_e}\Omega_e^2\tau_e^2\right)$. The z axis is parallel to the magnetic field and the x axis to the direction of decreasing density. Using the quasi-neutrality condition $n_e = n_i$, we can substitute (I.16) for n_e/n in (I.17), obtaining the following dispersion equation

$$\omega = -i\frac{b_i}{b_e}\frac{k_z^2 D_e+k_\perp^2 D_\perp+ik_z u-i\kappa k_y D_a(1+y)^{-1}(\Omega_e\tau_e)^{-1}}{b_i/b_e+(\Omega_e\tau_e)^{-2}+k_z^2/k_\perp^2-ik_y\kappa/\Omega_e\tau_e k_\perp^2} \tag{I. 18}$$

where $\kappa = -\frac{d\ln n}{dx}$. According to (I.18) the instability condition $\operatorname{Im}\omega > 0$ can be represented in the following form:

$$X^4+(2+y)X^2+1+y+\frac{\kappa^2 k_y^2}{k^4}\frac{y}{1+y} < \frac{b_e}{b_i}y\frac{uk_y\kappa}{D_e k^3}X \tag{I. 19}$$

where $X = k_z\Omega_e\tau_e/k_\perp$.

According to (I.19) the instability occurs only in the presence of a longitudinal current, $u \neq 0$. Since the right hand side of (I.19) increases when k decreases, a perturbation must develop in the first place with the minimum possible k, i.e. $k \sim \kappa$. The corresponding perturbation has the form of a curvature of the filament as a whole. Since usually $b_e/b_i \sim 10^2-10^3 \gg 1$ and $u\kappa/D_e \sim 1$, the instability may arise even for $y < 1$. In this case $X \sim 1$, i.e. $k_z \sim \kappa(\Omega_e\tau_e)^{-1} \ll \kappa$. In other words the longitudinal wavelength of the perturbation is considerably larger than the tube radius.

All these qualitative conclusions correspond adequately to the experimental data and more exact calculations lead to quantitative agreement with the experiment also. Figure 7, for instance, shows a comparison between the theoretical relationship between H_c/p and ap (a being the tube

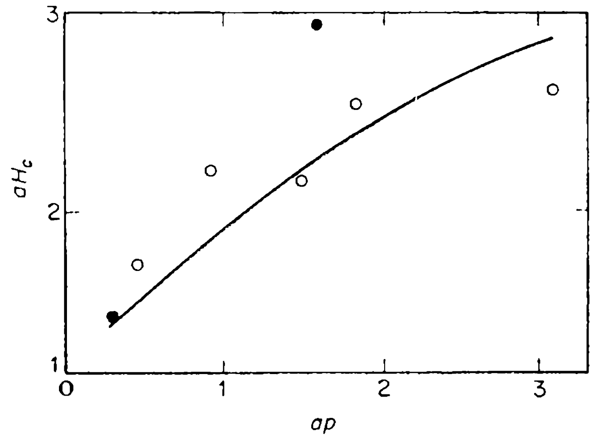

FIG. 7. Comparison of theoretical and experimental dependence of critical magnetic field on neutral gas pressure, discharge in helium

radius, p being neutral gas pressure) and experimental data of Lehnert and Hoh (32). The appearance of a helical curvature of the filament for $H > H_c$ was demonstrated directly by Allen *et al.* (34) using streak photography. Paulikas and Pyle (35) have published a detailed investigation of the transition through the critical field. Figure 8 shows their experimental relationship between the pitch of the twisted filament and the frequency of its rotation,

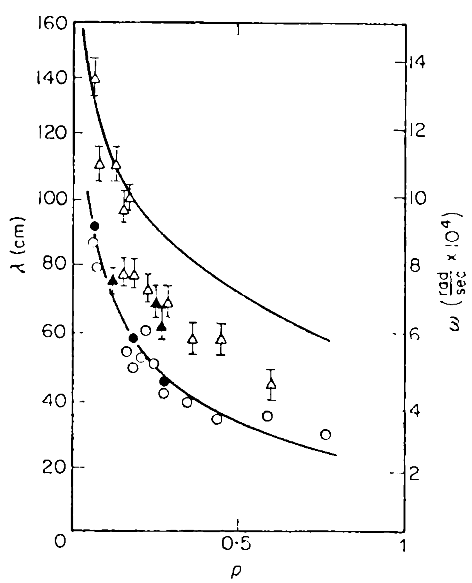

FIG. 8. Comparison of theoretical and experimental dependence of pitch of helical discharge λ and frequency of rotation ω on neutral gas pressure (discharge in helium)

and the neutral gas pressure for a discharge in helium. The theoretical relationship is shown by the full lines. As may be seen, the agreement between theory and experiment is very satisfactory.

The helical distortion of the filament for a magnetic field slightly exceeding H_c is simply the result of the convection of the plasma. For small $H-H_c$ it is again possible to use the quasi-linear approximation to determine the convective loss of the plasma and the amplitude of the distortion. Such a calculation (33) shows that the outward flux of plasma increases with $H-H_c$ and the amplitude increases as $\sqrt{H-H_c}$. The curved section 2 on Fig. 6 represents the dependence of E on H calculated from this theory. As may be seen, for a magnetic field only slightly greater than the critical field, the theoretical and experimental dependence agree very well. For H $\gg H_c$ the positive column goes over into a turbulent state which will be considered in Section (IV.4).

3. Quasi-linear Approximation in Kinetics

We now turn to the opposite extreme of a collisionless plasma and consider first the application of the quasi-linear method to the excitation of Langmuir oscillations by an electron beam (7) (8) (9).

(a) *Electron Beam in a Plasma*

Suppose that the electron distribution function has the form shown in Fig. 9 by the full line. The second maximum represents a diffuse beam of

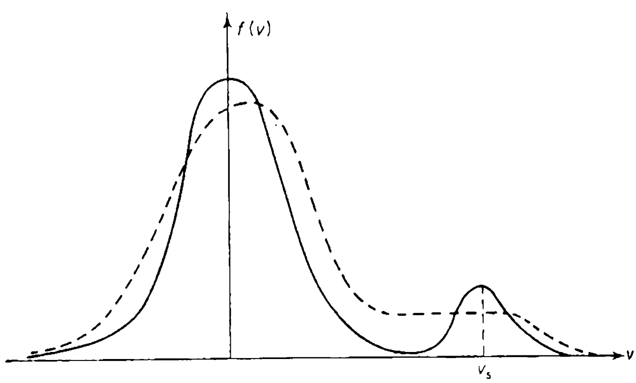

Fig. 9. Initial (full curve) and final (dashed curve) distribution functions during excitation of Langmuir oscillations

electrons of mean velocity V_s superimposed on the main group of thermal electrons.

As we know, such a velocity distribution is unstable: Langmuir waves with phase velocities in the region where $\frac{df}{dv} > 0$ will increase with time, since the number of electrons overtaking a wave and transferring energy to it will be larger than the number of electrons withdrawing energy from it.

Thus, as a result of Landau's inverse damping mechanism a group of waves will be built up, the phase velocities of which cover the whole interval where $\frac{df}{dv} > 0$.

If the number of resonance particles is small, i.e. the particle density in the beam is considerably smaller than the density of the thermal electrons, the growth rate of the waves γ will be considerably smaller than the frequency ω. For $\gamma/\omega \ll 1$ the interaction between the waves can be neglected and we can use the quasi-linear approximation, in which the only non-linear effect considered is the reaction of the oscillations on the " background "—the averaged distribution function.

In the unidimensional case the motion of the electrons will be described by the following system of equations:

$$\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x} - \frac{eE}{m}\frac{\partial F}{\partial v} = 0 \tag{I. 20}$$

$$\frac{\partial E}{\partial x} = 4\pi e\left\{\int F\,dv - n\right\} \tag{I. 21}$$

where n is the ion density and F the electron distribution function.

Let us split up the distribution function into two parts: $F = f + f^{(1)}$ where f is the mean distribution, regarded as a slowly varying function of t, and f is the oscillating part which averages out to zero. The function $f^{(1)}$ represents a system of oscillations with randomly distributed phases

$$f^{(1)} = \int f_k e^{i\omega_k t}\,dk \tag{I. 22}$$

where ω_k is the characteristic frequency of the k^{th} mode. For the Langmuir oscillations $\omega_k \simeq \omega_0 + i\gamma_k$, where $\omega_0 = \sqrt{\frac{4\pi e^2 n}{m}}$, $\gamma_k = \frac{\pi\omega_0^3}{2k^2 n}\frac{\partial f}{\partial v}\bigg|_{v=\omega/k}$.

From the linearised form of eqn. (I.20), we determine the relation between f_k and the amplitude of the electric field E_k in the k^{th} mode

$$f_k = \frac{e}{m}\frac{i}{\omega_k - kv}\frac{\partial f}{\partial v}E_k + A_k\delta(\omega_k - kv) \tag{I. 23}$$

Here the second term allows for the possibility of the injection of weakly modulated beams into the plasma. For simplicity we assume that such beams are absent, i.e. $A_k = 0$.

The equation for the averaged function f is obtained by writing $F = f + f^{(1)}$ in equation (I.20) and expressing $f^{(1)}$ in terms of E_k from (I.22) and (I.23). Averaging with respect to the statistical ensemble, i.e. with respect to the random phase, and noting that $\frac{df}{\partial x} = 0$, $E_k^* = E_{-k}$, we obtain

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\left(D_v\frac{\partial f}{\partial v}\right) \tag{I. 24}$$

where

$$D_v = \frac{e^2}{m^2}\int \frac{\gamma_k E_k^2(t)}{(\omega_k - kv)^2 + \gamma_k^2}\,dk \tag{I.25}$$

For a constant mean distribution function the dependence of E_k^2 on t would be simply exponential, $E_k^2 \sim e^{2\gamma_k t}$, but for a weak dependence of f on time E_k^2 is given by

$$\frac{dE_k^2}{dt} = 2\gamma_k E_k^2 \tag{I. 26}$$

Equations (I.24) and (I.26) are the basic equations of quasi-linear kinetic theory.

With the aid of these equations we can now examine the development of unstable oscillations of a plasma. Let us consider first the group of resonance electrons. Since $\gamma \ll \omega$, for this group of particles the coefficient of diffusion in velocity space D_v can be written sufficiently accurately in the following form

$$D_v = \frac{\pi e^2}{m^2}\int \delta(\omega_k - kv)E_k^2\,dk \tag{I. 27}$$

This coefficient of diffusion is different from zero only in the region of velocities where $\gamma > 0$, i.e. $\frac{\partial f}{\partial v} > 0$. Because of the diffusion the distribution function will be flattened in this region until the growth rate vanishes, i.e. until a " plateau " appears in the distribution function. At this time a stationary spectrum of supra-thermal oscillations will be established in the plasma. To determine the spectrum we use eqns. (I.24), (I.26), (I.27) and the relation $\gamma_k = \frac{\pi\omega_0^3}{2k^2 n}$ to obtain the following equation

$$\frac{\partial}{\partial t}\left\{f + \frac{\partial}{\partial v}\left(\frac{e^2 n}{m^2\omega_0 v^3}\,E_v^2\right)\right\} = 0$$

whence we obtain

$$f + \frac{\partial}{\partial v}\left(\frac{e^2 n}{m^2\omega_0 v^3}\,E_v^2\right) = \text{const}$$

where instead of k we use the variable $v = \omega_0/k$. From this relation we determine the oscillation amplitude in the steady state

$$E_v^2(t=\infty) = \frac{m^2\omega_0}{e^2 n}v^3\int_{v_1}^{v}\{f(t=\infty) - f(t=0)\}\,dv \tag{I. 28}$$

where $f(t = \infty) = \text{const}$. In (I.28) the value $v = v_1$ represents the lower boundary of the region of velocities where the plateau is established. It is determined by the condition of the conservation of the total number of resonance particles, namely $\int f dv = \text{const}$.

Let us now consider what happens to the non-resonance thermal electrons. For such electrons $(\omega - kv)^2 \gg \gamma^2$ and, consequently, the corresponding coefficient of diffusion, broadly speaking, is ω/γ times smaller than the coefficient of diffusion for the resonance particles. But since the number of thermal electrons is just ω/γ times greater than that of resonance electrons, their diffusion cannot be neglected. From (I.25) the corresponding diffusion coefficient can be represented approximately in the following form:

$$D_v = \frac{e^2}{m^2} \int \frac{\gamma_k E_k^2}{(\omega - kv)^2} dk \qquad (\text{I. } 29)$$

Since the coefficient of diffusion of the non-resonance particles is small, we can replace f by $f(t = 0)$ in this region on the right hand side of eqn. (I.24). Considering (I.26), the integration of (I.24) with respect to time is straightforward and we obtain the following expression for the change in the distribution function of the thermal (non-resonance) electrons:

$$f(t = \infty) - f(t = 0) = \frac{e^2}{2m^2} \int E_k^2(t = \infty) \frac{\partial}{\partial v} \left[\frac{1}{(\omega - kv)^2} \frac{\partial f}{\partial v} \right] dk$$

Thus, the full distribution function for $t = \infty$ will have the form represented on Fig. 9 by the dotted graph. All the momentum and half of the energy lost by the beam in building up the oscillations are transferred to the thermal electrons, which leads to a distortion of their distribution function. The remainder of the energy lost by the beam is stored in thc electric field.

We can show for instance that the total momentum of the beam and of the plasma are conserved. By multiplying eqn. (I.24) by mv and integrating with respect to v we obtain

$$\frac{d}{dt} \int mvf\, dv = -\frac{\pi e^2}{m} \int \delta(\omega_0 - kv) E_k^2 \frac{\partial f}{\partial v} dk\, dv - \frac{e^2}{m} \int \frac{\gamma_k}{(\omega_0 - kv)^2} \frac{\partial f}{\partial v} E_k^2\, dk\, dv$$

where in the second integral the range of integration covers the region of the thermal electrons $|v| \sim v_e = \sqrt{\dfrac{2T_e}{m}}$.

Since in this region $(\omega_0 - kv)^{-2} \approx \dfrac{1}{\omega_0^2} + \dfrac{2kv}{\omega_0^3}$, we can integrate with respect to v and, substituting for γ_k its expression given above (following (I.22)), it can be seen that the integrals on the right hand side of this expression exactly compensate one another. Hence the total momentum is conserved.

(b) *Waves in a Plasma*

In the general case the problem of describing a weakly turbulent state in the quasi-linear approximation can be split into two parts: the determination of the wave field and the consideration of its reaction back on the particles. Let us consider the first of these problems in greater detail. Since in practice the plasma is almost always inhomogeneous, we must understand the propagation of waves in an inhomogeneous plasma. Since we have in

mind a turbulent plasma in which a large number of waves has been excited, the wavelength for the main part of the oscillations must be considerably smaller than the characteristic dimensions and consequently the inhomogeneity can be considered weak.

We recall that in a homogeneous unbounded plasma all characteristic oscillations are plane waves of the form exp $(-i\omega t + i\mathbf{kr})$. Maxwell's equations for such oscillations have the following form:

$$[\mathbf{kH}_{\mathbf{k}\omega}] + \frac{\omega}{c}\hat{\varepsilon}\mathbf{E}_{\mathbf{k}\omega} = 0 \tag{I. 30}$$

$$[\mathbf{kE}_{\mathbf{k}\omega}] - \frac{\omega}{c}\mathbf{H}_{\mathbf{k}\omega} = 0 \tag{I. 31}$$

where $\hat{\varepsilon}$ is the dielectric permeability tensor of the plasma.

Expressing **H** in terms of **E** from (I.31), we obtain a single vector equation for **E**, which in Cartesian co-ordinates has the following form:

$$\sum_{\beta}\left(k^2\delta_{\alpha\beta} - \frac{\omega^2}{c^2}\varepsilon_{\alpha\beta} - k_\alpha k_\beta\right)E_\beta(\mathbf{k},\omega) = 0 \tag{I. 32}$$

so that the frequency ω of the characteristic oscillations must satisfy the dispersion equation

$$D(\mathbf{k},\omega) = Det\,\left\|k^2\delta_{\alpha\beta} - \omega^2c^{-2}\varepsilon_{\alpha\beta} - k_\alpha k_\beta\right\| = 0 \tag{I. 33}$$

In general the characteristic frequencies are complex: from the point of view of the quasi-linear approximation we are interested mainly in the special case in which the imaginary part of the frequency is small compared to the real part and is due to the interaction of the wave with resonance particles. The growth rate of the wave is determined by the anti-Hermitian part of the dielectric permeability $i\varepsilon''_{\alpha\beta} = \dfrac{\varepsilon_{\alpha\beta} - \varepsilon^*_{\beta\alpha}}{2}$ which is small compared with the Hermitian part $\varepsilon'_{\alpha\beta} = \dfrac{\varepsilon_{\alpha\beta} + \varepsilon^*_{\beta\alpha}}{2}$.

If $\varepsilon''_{\alpha\beta}$ is neglected the characteristic functions represent a plane wave of constant amplitude. When these waves are uniformly distributed in space and are statistically independent, the correlation function for the electric field has the following form

$$\langle E_\alpha(\mathbf{k},\omega)E^*_\beta(\mathbf{k}',\omega')\rangle = \sum_i a^i_\alpha a^{i*}_\beta I_i(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega')\delta(\omega-\omega^i_{\mathbf{k}}) \tag{I. 34}$$

where the summation is over all characteristic frequencies $\omega^i_{\mathbf{k}}$ belonging to the wave vector **k**, $\mathbf{a}_i$ is the unit polarisation vector (which is in general complex) defined by the relation **E** = **a**E, and I(k) is the spectral distribution of the electric field.

In the presence of a magnetic field, several different characteristic frequencies correspond to one value of **k** in general. In this case all waves can be considered uncorrelated, in contrast to conventional optics where it is necessary to consider the correlation of the polarisations, and for a complete

description of two transverse oscillations it is necessary to introduce the four Stokes parameters.

When the small anti-Hermitian part $\varepsilon''_{\alpha\beta}$ is taken into consideration, the amplitude of the oscillations varies with time and the correlation function will not be simply proportional to $\delta(\omega-\omega')$. But when considering a large number of waves in a wide frequency interval, it is still possible to use the relation (I.34) as an approximation, by allowing $I(\mathbf{k})$ to be a slowly varying function of time. Differentiating this relation with respect to time gives

$$-i(\omega-\omega')\langle E_\alpha(\mathbf{k},\omega)E^*_\beta(\mathbf{k}',\omega')\rangle = \\ = \sum_i \frac{\partial}{\partial t}(a^i_\alpha a^{i*}_\beta I_i(\mathbf{k},t))\delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega')\delta(\omega-\omega^i_{\mathbf{k}}) \qquad \text{(I. 35)}$$

from which it is immediately apparent that the variation of I with time leads to a " broadening " of the δ function of $\omega-\omega'$ in the expression for the correlation function.

A spatial inhomogeneity must lead to an analogous broadening of the function $\delta(\mathbf{k}-\mathbf{k}')$, and this can be considered approximately by allowing a weak dependence of I on $\mathbf{r}$. In this case the operator ∇ corresponds to $i(\mathbf{k}-\mathbf{k}')$.

The approximate representation of the correlation function in the form (I.34) corresponds physically to the concept that the electric field of the oscillations forms a set of statistically independent wave packets. If the characteristic wavelength of the oscillations is considerably smaller than the dimensions of these packets, then over limited intervals of time we can neglect their dispersive spreading. To obtain an equation describing the behaviour of such packets we use the following averaging operation: we multiply eqn. (I.30) by $\mathbf{E}^*_{\mathbf{k}'\omega'}$, eqn. (I.31) by $\mathbf{H}^*_{\mathbf{k}'\omega'}$, subtract the one from the other and then anti-symmetrise the resulting relation, i.e. subtract from it an analogous complex conjugate relation with $\mathbf{k}$, ω and $\mathbf{k}'$, ω' transposed. In the relation so obtained the value of the quantity $\varepsilon'_{\alpha\beta}(\mathbf{k},\omega)-\varepsilon'_{\alpha\beta}(\mathbf{k}',\omega')$ can be expanded in series for small $\mathbf{k}-\mathbf{k}'$ and $\omega-\omega'$. Finally we insert $-i\nabla$ and $i\frac{\partial}{\partial t}$ respectively for $\mathbf{k}-\mathbf{k}'$ and $\omega-\omega'$. An analogous exchange must be carried out in the other components where corresponding differences appear. For a homogeneous plasma we thus obtain the following energy balance equation

$$\frac{\partial W}{\partial t} + \operatorname{div}\mathbf{S} = -\frac{\omega}{4\pi}\operatorname{Im}\langle \mathbf{E}^*_{\mathbf{k}\omega}\varepsilon\mathbf{E}_{\mathbf{k}\omega}\rangle \qquad \text{(I. 36)}$$

where W is the energy of the $\mathbf{k}^{\text{th}}$ wave given by

$$W = \frac{1}{8\pi}\frac{\partial}{\partial\omega}\langle\omega\mathbf{E}^*\hat{\varepsilon}'\mathbf{E}\rangle + \frac{1}{8\pi}\langle\mathbf{H}^*\mathbf{H}\rangle \\ = \frac{1}{8\pi}\frac{\partial}{\partial\omega}\left\{\langle\omega\mathbf{E}^*\hat{\varepsilon}'\mathbf{E}\rangle - \frac{c^2k^2}{\omega}\langle\mathbf{E}^*\mathbf{E}\rangle + \frac{c^2}{\omega}\langle(\mathbf{k}\mathbf{E}^*)(\mathbf{k}\mathbf{E})\rangle\right\} \qquad \text{(I. 37)}$$

and $\mathbf{S}$ is the energy flux

$$\mathbf{S} = -\frac{1}{8\pi}\left\{\langle\omega\mathbf{E}^*\hat{\varepsilon}'\mathbf{E}\rangle - \frac{c^2k^2}{\omega}\langle\mathbf{E}^*\mathbf{E}\rangle + \frac{c^2}{\omega}\langle(\mathbf{kE}^*)(\mathbf{kE})\rangle\right\} \tag{I. 38}$$

$\hat{\varepsilon}'$ is the Hermitian part of the tensor $\hat{\varepsilon}$. In the expressions for W and $\mathbf{S}$ the differentiation with respect to ω and $\mathbf{k}$ can, in view of (I.32), be carried out at constant $\mathbf{E}^*$, $\mathbf{E}$.

Multiplying eqn. (I.32) by E_α and summing the result with respect to α, we obtain

$$G = \frac{\omega^2}{c^2}\mathbf{E}^*\hat{\varepsilon}\mathbf{E} - k^2\mathbf{E}^*\mathbf{E} + (\mathbf{kE}^*)(\mathbf{kE}) = 0 \tag{I. 39}$$

and it is clear that W and $\mathbf{S}$ are given by the derivatives of the mean value of the function G, defined by

$$\langle G\rangle = \left\{\frac{\omega^2}{c^2}(\mathbf{a}^*\hat{\varepsilon}\mathbf{a}) - k^2 + |\mathbf{ka}|^2\right\} I \equiv D_1 I \tag{I. 40}$$

The quantities D_1 and D (see (I.33)) vanish at the same frequency ω, so that at some other $\omega = \omega_\mathbf{k}$ they can be supposed proportional to one another. We then obtain

$$\gamma_\mathbf{k} = -\frac{\operatorname{Im} D_1}{\dfrac{\partial D_1}{\partial\omega}}; \qquad \mathbf{U}_\mathbf{k} = \frac{d\omega_\mathbf{k}}{d\mathbf{k}} = -\frac{\partial D_1}{\partial\mathbf{k}}\left(\frac{\partial D_1}{\partial\omega}\right)^{-1} \tag{I. 41}$$

where $\gamma_\mathbf{k}$ is the growth rate of the $\mathbf{k}^{\text{th}}$ wave, $\mathbf{U}_\mathbf{k}$ its group velocity. Using these relations we transform the energy balance eqn. (I.36) to the following simpler form

$$\frac{\partial I_\mathbf{k}}{\partial t} + \mathbf{U}_\mathbf{k}\frac{\partial I_\mathbf{k}}{\partial\mathbf{r}} = 2\gamma_\mathbf{k} I_\mathbf{k} \tag{I. 42}$$

Let us now consider an inhomogeneous plasma. The inhomogeneity leads to several additional features. Most important, in the tensor $\varepsilon_{\alpha\beta}$ some small additions appear which are proportional to the gradients of the mean distribution functions for the electrons and ions. In the presence of a magnetic field these additional terms represent the effect of the drift currents. Moreover, the permeability $\varepsilon_{\alpha\beta}$ becomes a slowly varying function of the co-ordinates and therefore, in the expression for the determinant $D(\mathbf{k}, \omega)$, $\varepsilon_{\alpha\beta}$ must be differentiated with respect to r wherever it appears multiplied by $\mathbf{k}$, which is equivalent to $-i\nabla$. Let us assume that all these corrections have been made; we retain the previous designation $D(\mathbf{k}, \omega)$ for the resulting determinant.

The other new feature in the propagation of the wave packets in an inhomogeneous plasma is related to a correlation of waves with wave vectors which are close to one another induced by the inhomogeneity. In a homogeneous plasma, the components of the tensor $\varepsilon_{\alpha\beta}$ in co-ordinate representation depend only on the distance between the point of observation $\mathbf{r}$, where the current is calculated, and the source point $\mathbf{r}'$, the field at which excites the

current. Therefore in the $\mathbf{k}$ representation the components $\varepsilon_{\alpha\beta}$ are simply numbers. In a weakly inhomogeneous plasma, however, $\varepsilon_{\alpha\beta}$ must be a function of both r and r', but can be represented in the form $\varepsilon_{\alpha\beta}\left(\mathbf{r}-\mathbf{r}', \frac{\mathbf{r}+\mathbf{r}'}{2}\right)$ where the dependence on the second argument is weak. Transforming to a Fourier representation, the dependence on the first argument changes into a dependence on $\mathbf{k}$, while in the case of the second argument we may expand about the point $\mathbf{r}$, where the wave packet is located. To first order we have in the co-ordinate representation

$$\varepsilon_{\alpha\beta}\left(\mathbf{r}-\mathbf{r}', \frac{\mathbf{r}+\mathbf{r}'}{2}\right) = \varepsilon_{\alpha\beta}(\mathbf{r}-\mathbf{r}', \mathbf{r}) + \frac{\mathbf{r}'-\mathbf{r}}{2} \frac{\partial}{\partial \mathbf{r}} \varepsilon_{\alpha\beta}(\mathbf{r}-\mathbf{r}', \mathbf{r})$$

where the differentiation of ε in the second component is performed only with respect to the second argument. Now transforming to the Fourier representation $\mathbf{r}'-\mathbf{r}$ goes over into $-i\frac{\partial}{\partial \mathbf{k}}$, and the second component will have the following form

$$-\frac{1}{2} \frac{\partial \varepsilon_{\alpha\beta}(\mathbf{k}, \omega, \mathbf{r})}{\partial \mathbf{r}} i \frac{\partial}{\partial \mathbf{k}}$$

Thus in the relationship between $\varepsilon_{\alpha\beta}$ and $\mathbf{r}$, t in eqn. (I.32) we must add some small terms proportional to the derivatives of the electric field with respect to $\mathbf{k}$ and ω. If we now repeat the derivation of the energy balance eqn. (I.36), we find that these additional terms lead to a term containing the derivative with respect to $\mathbf{k}$ of the spectral function. As a result eqn. (I.42) for a weakly inhomogeneous plasma assumes the following form (see refs. (12), (36a))

$$\frac{\partial I_{\mathbf{k}}}{\partial t} + \frac{\partial \omega_{\mathbf{k}}}{\partial \mathbf{k}} \frac{\partial I_{\mathbf{k}}}{\partial \mathbf{r}} - \frac{\partial \omega_{\mathbf{k}}}{\partial \mathbf{r}} \frac{\partial I_{\mathbf{k}}}{\partial \mathbf{k}} = 2\gamma_{\mathbf{k}} I_{\mathbf{k}} \tag{I. 43}$$

where $\omega_{\mathbf{k}} = \omega_{\mathbf{k}}(\mathbf{r}, t)$ is the characteristic frequency obtained from the dispersion relation $D(\mathbf{k}, \omega, \mathbf{r}, t) = 0$. Its derivatives with respect to $\mathbf{k}$ and $\mathbf{r}$ can be determined from the rule for the differentiation of implicit functions

$$\frac{\partial \omega_{\mathbf{k}}}{\partial \mathbf{k}} = -\frac{\partial D}{\partial \mathbf{k}} \Big/ \frac{\partial D}{\partial \omega}; \qquad \frac{\partial \omega_{\mathbf{k}}}{\partial \mathbf{r}} = -\frac{\partial D}{\partial \mathbf{r}} \Big/ \frac{\partial D}{\partial \omega} \tag{I. 44}$$

Note that for longitudinal oscillations the determinant D becomes simply the dielectric permeability ε.

Equation (I.43) has a simple physical meaning. The second term on the left describes the motion of the wave packet in space with a group velocity $\mathbf{U}_{\mathbf{k}} = \frac{\partial \omega_{\mathbf{k}}}{\partial \mathbf{k}}$, and the third shows that simultaneously the wave packet is distorted in such a way that the oscillation frequency remains constant.

Note that we have assumed throughout that the wave vector and the frequency $\omega_{\mathbf{k}}$ vary continuously. In fact the characteristic oscillation frequencies of an inhomogeneous plasma must be " quantised ", i.e. must pass

through a discrete series of values (see ref. (36)), but since the difference between adjacent characteristic frequencies of short wave perturbations is very small, this effect can be neglected.

It must be remembered that eqn. (I.43) corresponds to the zero order W.K.B. approximation. To this degree of accuracy it is completely irrelevant whether the term representing the divergence of the energy flux is written in the form shown in (I.43) or in the form div ($\mathbf{U}_\mathbf{k}\, I_\mathbf{k}$). When we consider a large number of modes covering a broad band of wave numbers this approximation is obviously quite sufficient.

From the point of view of the quasi-linear approximation, we are interested in the case in which $\gamma_\mathbf{k}$ is small. In this case the time derivative of $I_\mathbf{k}$ can also be considered small, so that in zero approximation we shall have

$$\frac{\partial\omega_\mathbf{k}}{\partial\mathbf{k}}\frac{\partial I_\mathbf{k}}{\partial\mathbf{r}} - \frac{\partial\omega_\mathbf{k}}{\partial\mathbf{r}}\frac{\partial I_\mathbf{k}}{\partial\mathbf{k}} = 0 \tag{I. 45}$$

It follows, therefore, that $I_\mathbf{k} = I\,(\omega_\mathbf{k})$ i.e. the dependence of $I_\mathbf{k}$ on $\mathbf{r}$ and $\mathbf{k}$ is mainly determined by the relationship of $I(\omega)$ and $\omega_\mathbf{k}(\mathbf{r})$. In other words, $I_\mathbf{k}$ represents a set of waves with different frequencies, the amplitude of each of these waves is independent of $\mathbf{r}$, and the corresponding wave vector $\mathbf{k} = \mathbf{k}\,(\mathbf{r})$ is defined by the relation $\omega_\mathbf{k} = \omega_s = \text{const.}$ In the next approximation we must consider the dependence of $I_\mathbf{k}$ on the time and a possible weak dependence on $\mathbf{r}$. For waves with frequency ω_s we shall have $I_\mathbf{k} = I_s\,(\mathbf{r}, t)\,\partial\,(\omega_\mathbf{k} - \omega_s)$. Substituting this expression in (I.43) gives

$$\frac{\partial I_s}{\partial t} + \mathbf{U}_s\frac{\partial I_s}{\partial\mathbf{r}} = 2\gamma_s I_s \tag{I. 46}$$

Consider for simplicity a one-dimensional case when the plasma is inhomogeneous along one co-ordinate only, say x. Suppose moreover that $\omega_\mathbf{k}$ is a symmetrical function of k_x, i.e. to each frequency correspond two waves propagating in opposite directions. Suppose further that the waves are reflected without absorption either from the walls or from the turning points (points at which $k_x = 0$).

For small γ and $\frac{\partial}{\partial t}$ the function I_s can, according to (I.46), be considered independent of x. Multiplying eqn. (I.46) by U_s^{-1} and integrating with respect to x, we eliminate the second term and obtain

$$\frac{\partial I_s}{\partial t} = 2\langle\gamma_s\rangle I_s \tag{I. 47}$$

where $\langle\gamma_s\rangle = \int\frac{\gamma_s(x)\,dx}{U_s(x)}\Big/\int\frac{dx}{U_s(x)}$ is the mean growth rate of the wave ω_s.

Clearly the stationary state will be reached when $< \gamma_s > \, = 0$, but for this to occur it is not necessary for $\gamma_s\,(x)$ to vanish everywhere. It is now trivial to include in this scheme absorption at the walls, and if this absorption is large enough we may have a steady state or even damping, even when the local growth rate is positive everywhere in the plasma.

(c) *Absolute Instability*

In the preceding section, we did not take into account the dispersive spreading of the wave packets. In several problems such an approximation is fully justified, but this is not always true. We shall consider here a specific example where the spreading of the wave packet must be taken into account.

In Section 3(a) we considered a beam instability in an unbounded plasma. However, in practice all experiments with beams except experiments with toroidal systems are conducted in bounded tubes. The problem therefore arises as to whether an instability can develop in such a tube. If we approach this problem from the point of view of geometrical optics, within the framework given in the preceding section on non-spreading wave packets, we ought to describe the build-up of perturbations with the aid of eqn. (I.43). According to this equation, in the absence of a feed-back between the entrance and exit of the beam, which can be realised, for example, by waves propagating against the beam (" backward " waves), any perturbation will move along the tube with a certain group velocity. The plasma and beam then operate as an amplifier rather than as an oscillator. Such an instability is called a convective instability, in contrast to an absolute instability where the perturbation grows at every point in space (refs. 25, 37–41).

We shall now show that the necessary feed-back can be produced by the dispersive spreading of the wave packet. We shall find that there is a bounding group velocity U_c such that for $U < U_c$ the instability is absolute, while for $U > U_c$ it is convective. We shall follow the treatment of ref. (41).

Suppose the growth rate γ_k as a function of k has a maximum at $k = k_0$. Obviously, after the lapse of a sufficiently long time, any given initial perturbation will be deformed so that it will have a sharp maximum at $k = k_0$. It is therefore sufficient to consider a wave packet consisting of plane waves with wave numbers close to k_0. It is then possible to make a series expansion of γ_k and ω_k with respect to the small difference $k-k_0$:

$$\gamma_k = \gamma_{k_0} - \frac{\alpha}{2}(k-k_0)^2, \qquad \omega_k = \omega_{k_0} + U(k-k_0) + \frac{\beta}{2}(k-k_0)^2 \quad \text{(I. 48)}$$

where $\alpha = \left.\dfrac{\partial^2\gamma_k}{\partial k^2}\right|_{k=k_0}$, $U = \left.\dfrac{\partial\omega_k}{\partial k}\right|_{k=k_0}$, $\beta = \left.\dfrac{\partial^2\omega_k}{\partial k^2}\right|_{k=k_0}$. We now transform to a moving co-ordinate system in which $\omega_{k0} = 0$. The evolution with time of the wave packet is then given by

$$E(x,t) = A_{k_0}\exp(\gamma_{k_0}t - ik_0x)$$
$$\int \exp\left\{i(x-Ut)(k-k_0) - \frac{(\alpha-i\beta)t}{2}(k-k_0)^2\right\}dk \quad \text{(I. 49)}$$

where A_{k0} is the initial amplitude of the electric field. Provided that the difference $x-Ut$ does not increase too rapidly with t, this integral can be evaluated and we obtain for the dependence of the electric field E on time the

following expression

$$E(x,t) = A_{k_0} \exp\left\{ -ik_0 x + \gamma_{k_0} t - \frac{(x-Ut)^2}{2(\alpha^2+\beta^2)t}(\alpha+i\beta) \right\}$$

where the last component under the exponent sign describes the spreading out of the wave packet. As may be seen, for

$$U^2 < 2\gamma_{k_0} \frac{\alpha^2+\beta^2}{\alpha} \equiv U_c^2 \tag{I. 50}$$

the amplitude of the wave packet will increase with time at each point $x = \text{const}$. In the opposite case $U > U_c$ the amplitude increases with time only in the moving system of co-ordinates.

Since (I.49) is accurate only near the maximum, i.e. not too far from the point $x = Ut$, the expression (I.50) for the critical value of the group velocity is only approximate.

(d) *Resonance and Adiabatic Interaction between Waves and Particles*

Let us now consider in greater detail the question of the interaction between waves and particles within the framework of the quasi-linear approximation. As we have already stated in Section 3(a), it is convenient to distinguish a resonance and an adiabatic interaction of particles with waves. Each separate particle has a resonant interaction with those waves whose phase velocity coincides with the particle velocity. During resonant interaction the electric field of the wave is constant in a system of co-ordinates related to the particle, and therefore during such interaction a considerable energy exchange takes place between the particle and the wave.

In the electric field of non-resonance waves, a particle performs oscillations with an amplitude governed by the electric field of the wave. A slow variation of the field amplitude leads to an adiabatic variation of the amplitude of the particle oscillations.

In the example 3(a) which we considered, the adiabatic interaction was described by a diffusion coefficient (I.29). In order better to visualise the effect of the adiabatic interaction and at the same time to clarify its representation in the form (I.29), we consider a simple example where the external field $Ee^{-i\omega t}$ with $k = 0$ is imposed on the plasma. For a slow variation of the amplitude E, according to (I.29) we obtain

$$\frac{\partial f}{\partial t} = \frac{e^2}{2\omega^2 m^2} \frac{\partial}{\partial v} \left(\frac{\partial E^2}{\partial t} \frac{\partial f}{\partial v} \right) \tag{I. 51}$$

Suppose that the square of amplitude of the field varies linearly $E^2 = E_0^2 \frac{t}{T}$ where $T \gg \omega^{-1}$. Equation (I.51) becomes a diffusion equation according to which the distribution function of the electrons, assumed cold at $t = 0$, will be at time $t = T$ given by

$$f = \frac{e^{-v^2/v_0^2}}{\sqrt{\pi} v_0}, \qquad v_0^2 = \frac{e^2 E^2}{m^2 \omega^2} \tag{I. 52}$$

An exact consideration would obviously give

$$f(v) = \langle \delta(v - v(t)) \rangle = \frac{\omega}{\pi |\dot{v}|} = \frac{1}{\pi \sqrt{v_0^2 - v^2}} \tag{I. 53}$$

The difference between (I.52) and (I.53) demonstrates the inaccuracy of the quasi-linear approximation when it is used for a single wave. When the number of modes of oscillation is increased the accuracy of (I.52) will improve, since the distribution function $f(v)$ as the probability density of the sum of a large number of random quantities must tend toward the Gaussian form.

For Langmuir oscillations, $\omega^2 = \omega_0^2 = \dfrac{4\pi e^2 n}{m}$, and the kinetic energy of the electrons $\dfrac{mnv_0^2}{2}$ is equal to the potential energy of the electric field $E^2/8\pi$. Thus, in the presence of an isotropic distribution of Langmuir oscillations the effective temperature T_{ef}, which defines the width of the average distribution function, is given by $T_{ef} = T_e + T_f$, where T_e is the true electron temperature, and $T_f = \dfrac{2}{3} \dfrac{\langle E^2 \rangle}{8\pi n}$. The value of T_f determines the kinetic energy of the oscillations and arises from the adiabatic interaction of the particles with the waves. Note that because $T_e > 0$, the effective temperature T_{ef} cannot be smaller than T_f.

The division of the interaction between particles and waves into resonance and adiabatic interaction becomes slightly more complex when going over to an inhomogeneous plasma. Let us for instance consider the simplest case of the Langmuir oscillations of an inhomogeneous plasma in the absence of a magnetic field.† We assume that the effect of the mean electric field $\mathbf{E}$ on the oscillations can be neglected. In the quasi-classical W.K.B. approximation the kinetic equation for the perturbed distribution function $f_{\mathbf{k}\omega}$ is then written in the following form

$$(-i\omega + i\mathbf{k}\mathbf{v}) f_{\mathbf{k}\omega} = \frac{e}{m} \left\{ \frac{\partial f}{\partial \mathbf{v}} \mathbf{E}_{\mathbf{k}\omega} + i \sum_{\alpha} \frac{\partial^2 f}{\partial \mathbf{v} \partial r_\alpha} \frac{\partial \mathbf{E}_{\mathbf{k}\omega}}{\partial k_\alpha} - i \frac{\partial^2 f}{\partial \mathbf{v} \partial t} \frac{\partial \mathbf{E}_{\mathbf{k}\omega}}{\partial \omega} \right\} \tag{I. 54}$$

where f is the averaged distribution function. In this equation the function $\dfrac{\partial f}{\partial v}$ and its derivatives with respect to $\mathbf{r}$ and t are considered independent of $\mathbf{r}$, t, and the distortion of the wave packet due to the inhomogeneity is included through the second and third terms in the curly brackets.

† In an inhomogeneous plasma the longitudinal waves are decoupled from the other modes only when the electric field of the wave acts parallel to the gradient of the unperturbed density. For any other direction the longitudinal oscillations are coupled with the transverse ones and the Langmuir oscillations must therefore be accompanied by electromagnetic radiation. However, the intensity of this radiation is small and the oscillations can still be considered approximately longitudinal.

The kinetic equation for the averaged distribution function has the following form

$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla f - \frac{e}{m}\mathbf{E}\frac{\partial f}{\partial \mathbf{v}} = S_{ef} \tag{I. 55}$$

where the term S_{ef}, which represents the collisions between electrons and waves, is given by the relation

$$S_{ef} = \frac{e}{m}\frac{\partial}{\partial \mathbf{v}}\int\int \langle \mathbf{E}^*_{\mathbf{k}'\omega'} f_{\mathbf{k}\omega}\rangle \, d\mathbf{k}\, d\omega\, d\mathbf{k}'\, d\omega' \tag{I. 56}$$

The expression underneath the integral sign in (I.56) can be written in the symmetrical form $\frac{1}{2}\langle \mathbf{E}^*_{\mathbf{k}'\omega'} f_{\mathbf{k}\omega} + \mathbf{E}_{\mathbf{k}\omega} f^*_{\mathbf{k}'\omega'}\rangle$ and we can then substitute for the function $f_{\mathbf{k}\omega}$ its expression in terms of $\mathbf{E}_{\mathbf{k}\omega}$ from (I.54). The contribution due to the first term in the right hand side of (I.54) is proportional to

$$\frac{1}{2}\left(\frac{i}{\omega - \mathbf{k}\mathbf{v} + i\nu} - \frac{i}{\omega' - \mathbf{k}'\mathbf{v} - i\nu}\right)\langle \mathbf{E}^*_{\mathbf{k}'\omega'} \mathbf{E}_{\mathbf{k}\omega}\rangle \tag{I. 57}$$

Bearing in mind that the correlation of the electric field approximates to a delta function, we can replace the real part in the round brackets in (I.57) by $2\pi\delta\,(\omega - \mathbf{k}\mathbf{v})$, and the imaginary part can be represented in the form

$$\left[i(\omega - \omega')\frac{\partial}{\partial \omega} + i(\mathbf{k} - \mathbf{k}')\frac{\partial}{\partial \mathbf{k}}\right](\omega - \mathbf{k}\mathbf{v})^{-1} = \left[-\frac{\partial}{\partial t}\frac{\partial}{\partial \omega} + \nabla\frac{\partial}{\partial \mathbf{k}}\right](\omega - \mathbf{k}\mathbf{v})^{-1} \tag{I. 58}$$

The last two terms in the curly brackets in (I.54) lead to terms which are proportional to the derivatives of $I_{\mathbf{k}\omega}$ with respect to $\mathbf{k}$ and ω. Since $\mathbf{E}_{\mathbf{k}} = \frac{\mathbf{k}}{k}E_{\mathbf{k}}$ we finally obtain

$$\begin{aligned} S_{ef} = {} & \frac{\pi e^2}{m^2}\frac{\partial}{\partial \mathbf{v}}\int \mathbf{k}\left(\frac{\mathbf{k}}{k^2}\frac{\partial f}{\partial \mathbf{v}}\right)\delta(\omega - \mathbf{k}\mathbf{v}) I_{\mathbf{k}\omega}\, d\mathbf{k}\, d\omega - \\ & - \frac{e^2}{2m^2}\frac{\partial}{\partial \mathbf{v}}\int \frac{\frac{\mathbf{k}}{k^2}\frac{\partial f}{\partial \mathbf{v}}}{\omega - \mathbf{k}\mathbf{v}}\left\{\nabla I_{\mathbf{k}\omega} - \frac{2\mathbf{k}}{k^2}(\mathbf{k}\nabla I_{\mathbf{k}\omega})\right\} d\mathbf{k}\, d\omega - \\ & - \frac{e^2}{2m^2}\frac{\partial}{\partial \mathbf{v}}\int \mathbf{k}\left\{\frac{\partial I_{\mathbf{k}\omega}}{\partial t}\frac{\partial}{\partial \omega} - \frac{\partial I_{\mathbf{k}\omega}}{\partial \omega}\frac{\partial}{\partial t} - \nabla I_{\mathbf{k}\omega}\frac{\partial}{\partial \mathbf{k}} + \frac{\partial I_{\mathbf{k}\omega}}{\partial \mathbf{k}}\nabla\right\}\frac{\frac{\mathbf{k}}{k^2}\frac{\partial f}{\partial v}}{\omega - \mathbf{k}\mathbf{v}}\, d\mathbf{k}\, d\omega \end{aligned} \tag{I. 59}$$

where the singularity $(\omega - \mathbf{k}\mathbf{v})^{-1}$ is integrated in the sense of its principal value. We have omitted a small term proportional to $I_{\mathbf{k}\omega}$ (i.e. not containing a derivative of $I_{\mathbf{k}\omega}$) because within the framework of our adopted zero order W.K.B. approximation we also neglected similar additions in the transfer eqn. (I.43). In (I.59) the first integral corresponds to the resonance interaction, the other two to the adiabatic interaction between the particles and the waves.

Multiplying (I.59) by $m\mathbf{v}$ and integrating the result with respect to $\mathbf{v}$, we obtain the force $\mathbf{F}$ acting on the plasma due to the interaction with the waves. Since for longitudinal oscillations $\varepsilon = 0$ we obtain

$$\mathbf{F} = -\frac{1}{8\pi}\int\left[\nabla I - \frac{2\mathbf{k}}{k^2}(\mathbf{k}\nabla I)\right] d\mathbf{k}\, d\omega +$$
$$+\frac{1}{8\pi}\int \mathbf{k}\left\{\frac{\partial I}{\partial t}\frac{\partial \varepsilon}{\partial \omega} - \frac{\partial I}{\partial \omega}\frac{\partial \varepsilon}{\partial t} - \frac{\partial I}{\partial \mathbf{r}}\frac{\partial \varepsilon}{\partial \mathbf{k}} + \frac{\partial I}{\partial \mathbf{k}}\frac{\partial \varepsilon}{\partial \mathbf{r}} + 2\,\mathrm{Im}\,\varepsilon I\right\} d\mathbf{k}\, d\omega \quad \text{(I. 60)}$$

where ε is the dielectric permeability of the plasma given by

$$\varepsilon = 1 + \frac{4\pi e^2}{mk^2}\int \frac{\mathbf{k}\dfrac{\partial f}{\partial \mathbf{v}}}{\omega - \mathbf{k}\mathbf{v}}\, d\mathbf{v} \quad \text{(I. 61)}$$

According to (I.43) the second integral in (I.60) vanishes and consequently the force $\mathbf{F}$ reduces to the divergence of the Maxwellian stresses. The second part of the force, related to the kinetic energy of the oscillations, is obtained from the averaged function and, according to (I.52), can be represented in the form of a gradient of the pressure tensor

$$p_{ij}^f = \frac{e^2 n}{m}\int \frac{k_i k_j}{k^2\omega^2} I_{\mathbf{k}\omega}\, d\mathbf{k}\, d\omega = \frac{1}{4\pi}\int \frac{k_i k_j}{k^2} I_{\mathbf{k}\omega}\, d\mathbf{k}\, d\omega \quad \text{(I. 62)}$$

where we have replaced ω^2 by $\omega_0^2 = \dfrac{4\pi e^2 n}{m}$.

The total force exerted on the plasma by the Langmuir oscillations is thus simply the negative gradient of the energy density of the electric field.

Consideration of the adiabatic interaction of the particles with the waves is important in other cases when the energy and momentum balance in a plasma supporting oscillations is discussed; in particular this interaction must be included when discussing the " anomalous " diffusion of a plasma.

It must be remembered that the division of the total interaction into resonant and an adiabatic parts is only possible for sufficiently small growth rates when one can refer to near-periodic oscillations. The condition $\gamma \ll \max(\omega, kv_T)$ must be fulfilled and this is also a necessary condition for the applicability of the quasi-linear approximation (v_T is the thermal velocity).

(e) *Enhanced Diffusion of a Plasma*

The expression for the collision term (I.59) can be generalised without difficulty to include longitudinal oscillations of a plasma in a magnetic field. For a two-component plasma an expression of the type of (I.59) must be written down for each component. The force $\mathbf{F}_j$ acting on the component j can then be represented in the form $\mathbf{F}_j = \dfrac{\mathbf{F}_0}{2} \pm \mathbf{F}_{ie}$ where $\mathbf{F}_0$ is the total force acting on the plasma as a whole and $\mathbf{F}_{ie}$ is the " frictional " force between the electrons and ions which is transmitted through the oscillations. By using a relation of the form of (I.60), it is not difficult to show that the

total force $\mathbf{F}_0$ is the divergence of the Maxwell stress tensor. This force can only lead to a macroscopic motion of the total plasma, including the frozen-in magnetic field. The diffusion across the magnetic field is determined by the frictional force between the electrons and ions, $\mathbf{F}_{ie}$, or more precisely its component across the density gradient. This conclusion follows immediately from the hydrodynamic equations of motion for each of the components which, as we know, describe fairly accurately the slow mean motion of the particles across a magnetic field.

This result is very important and must always be borne in mind when investigating the possibility that some given oscillations affect the diffusion of a plasma. In particular, it follows that high frequency oscillations, in which the ion motion can be neglected, cannot lead to diffusion of the plasma. At first sight this assertion might seem improbable, since each separate electron in such a field performs random motions which may be considered as diffusion. The corresponding coefficient of diffusion for a separate particle can be determined either by calculating the mean-square of the displacement (Taylor (42)) or from quasi-linear theory (43); for resonance particles this diffusion is described by the first term in (I.59). But if together with the diffusion of the resonance particles we also consider the displacement of the remaining electrons due to the adiabatic interaction, we obtain an expression of the type of (I.60), according to which the total current vanishes identically. Thus, the interaction of electrons with waves, if the ions are stationary, leads only to a diffusion of separate particles, i.e. to effects such as an intensified thermal conduction, rather than to a diffusion of the plasma as a whole. The interactions of these oscillations are therefore equivalent to electron–electron collisions.

The neglect of this important conclusion in papers dealing with enhanced diffusion has often led to erroneous statements. For instance, in some experimental papers attempts are made to relate the anomalous diffusion directly to the high frequency oscillations in which it is known that ions cannot participate. Theoretical considerations of enhanced diffusion are sometimes limited to only one component, for example, the electrons; in this case the problem of the mechanism making the diffusion ambipolar needs further consideration. This problem is automatically resolved when we determine the diffusive loss directly from the frictional force between the electrons and the ions

A similar error is incurred by not taking into account the " dragging along " of the waves by the particles, as a result of their interaction. For instance, in Taylor's paper (42) quoted above where Langevin's equation is used for the investigation of the random motion of the particles, a rather vague assumption was made that the oscillations are isotropic in the laboratory system of co-ordinates. This assumption, which is equivalent to assuming a strong coupling between the oscillations and the walls surrounding the plasma, led that author to the erroneous conclusion that the coefficient of enhanced diffusion cannot exceed Bohm's value. A similar error was also committed in ref. (44) where the coefficient of diffusion of the electrons was

calculated on the basis of the thermal cyclotron oscillations, which were considered isotropic in the laboratory system of co-ordinates, rather than in the system of co-ordinates moving with the electrons (this error has been corrected in a later paper by the same authors (45)).

In a complete discussion of the diffusion problem all these paradoxes are automatically resolved, and complete clarity, as we have seen, can only be achieved when the adiabatic interaction between the particles and the waves is considered. In those cases in which resonance interaction is impossible, the enhanced diffusion can be determined from the adiabatic interaction alone. As an example, we may quote the problem of the diffusion of a plasma due to drift waves excited by an external source, treated by Petviashvili (46), in which the electrons diffuse due to the resonant interaction with the waves, and the ions due to the adiabatic interaction.

II. INTERACTION BETWEEN WAVES IN WEAK TURBULENCE

THE quasi-linear method investigated in the preceding chapter has a very limited field of application, being suitable only for the description of states which are so weakly excited that strictly speaking they ought not to be called turbulent, since the most important property of turbulence—the non-linear interaction between the oscillations—is not yet apparent. In real turbulent processes in a plasma, the interaction between the oscillations generally plays an important part. The consideration of this interaction is the subject of this and the following chapters.

1. KINETIC WAVE EQUATION

(a) *Derivation of the Kinetic Wave Equation*

We are considering here only states where the interaction between the waves can be considered weak. This situation can be expected when the growth rate of the unstable perturbations is sufficiently small, i.e. $\gamma/\omega \ll 1$. This condition as it stands, however, may be misleading because the value of ω depends on the choice of the co-ordinates system. For instance, in the case of a set of sound waves travelling in the same direction, say parallel to the z axis, the frequencies of all the waves vanish when we transform to a co-ordinate system moving with the sound velocity. Even for a very small growth rate this must be considered a case of strong turbulence. However, for the same sound waves distributed isotropically, the weak interaction approximation can be used. Thus, although in what follows we shall use the condition $\gamma/\omega \ll 1$ as the weak coupling criterion, we must remember that the condition would be expressed more accurately by the statement that the growth time of the perturbations must be considerably larger than the characteristic time of conservation of the relative phases of the different waves.

Let us first consider oscillations in which the resonance interaction with particles plays no part. This would occur, for example, in the case of a cold plasma where the thermal motion of the particles can be neglected, and the magneto-hydrodynamic equations can be used to describe the motion of the plasma. Now a whole series of such problems lead to equations whose basic structure is similar, and can be represented in Fourier space by the following general scalar equation

$$(\omega-\omega_{\mathbf{k}}-i\gamma_{\mathbf{k}})C_{\mathbf{k}\omega} = \int V_{\mathbf{k}\omega,\,\mathbf{k}'\omega'}\,C_{\mathbf{k}'\omega'}\,C_{\mathbf{k}-\mathbf{k}',\,\omega-\omega'}\,d\mathbf{k}'\,d\omega' \qquad \text{(II. 1)}$$

We shall use (II.1) as a model equation to explore the character of the

interaction between the oscillations. The expression on the left hand side of eqn. (II.1) represents the linear part of the equation of motion and the expression on the right represents the Fourier transform of the quadratic terms. These quadratic terms describe the interaction between the various harmonics and the quantity of $V_{\mathbf{k}\omega,\,\mathbf{k}'\omega'}$ represents the matrix element of this interaction.

Note that since $C_{\mathbf{k}\omega}$ represents the Fourier transform of a real function, $C^*_{\mathbf{k}\omega} = C_{-\mathbf{k},\,-\omega}$. Using this relation in the complex conjugate of eqn. (II.1), changing the signs of $\mathbf{k}$, ω and $\mathbf{k}'$, ω' and noting that $\omega_{-\mathbf{k}} = -\omega_{\mathbf{k}}$, $\gamma_{-\mathbf{k}} = \gamma_{\mathbf{k}}$, we obtain an equation analogous to (II.1), but with a different matrix element. Comparing these equations we obtain

$$V^*_{\mathbf{k}\omega,\,\mathbf{k}'\omega'} = -\,V_{-\mathbf{k},\,-\omega;\,-\mathbf{k}',\,-\omega'} \tag{II. 2}$$

If the growth rate $\gamma_{\mathbf{k}} > 0$, small perturbations will increase with time until the non-linear interaction comes into play. In conditions approximating to equilibrium, the right hand side of (II.1) must be considered small, since in zero approximation we have

$$(\omega-\omega_{\mathbf{k}})C^{(0)}_{\mathbf{k}\omega} = 0 \tag{II. 3}$$

If the oscillations developed from random thermal motions, the individual modes can be considered completely independent. Multiplying equation (II.3) by $C^{(0)*}_{\mathbf{k}'\omega'}$ and averaging the result with respect to the statistical ensemble, i.e. with respect to the random phases of the separate oscillations, we obtain for a stationary and spatially homogeneous system of oscillations

$$\langle C^*_{\mathbf{k}'\omega'} C_{\mathbf{k}\omega}\rangle_0 = I_{\mathbf{k}}\delta(\omega-\omega_{\mathbf{k}})\delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega') \tag{II. 4}$$

If we replace $C_{\mathbf{k}\omega}$ by $C^{(0)}_{\mathbf{k}\omega}$ on the right hand side of (II.1), the non-linear term will play the part of an inducing force. The amplitude of the induced oscillations will be denoted by $C^{(1)}_{\mathbf{k}\omega}$. The presence of induced oscillations does not lead directly to damping of the waves, which becomes evident only in higher approximations. It follows, therefore, that the quantity CV must be of order $\sqrt{\gamma/\omega}$, and therefore in the equation for $C^{(1)}_{\mathbf{k}\omega}$ the growth rate $\gamma_{\mathbf{k}}$ can be omitted. Moreover, since the beats must attenuate more rapidly than the main oscillations, it is necessary to add to the right hand side of equation (II.1) for the beats in place of $-i\gamma_{\mathbf{k}}$ a small term $i\nu$, which represents the damping due to the higher correlations. We thus obtain the following equation for the first approximation

$$C^{(1)}_{\mathbf{k}\omega} = (\omega-\omega_{\mathbf{k}}+i\nu)^{-1}\int V_{\mathbf{k}\omega,\,\mathbf{k}'\omega'}C^{(0)}_{\mathbf{k}'\omega'}C^{(0)}_{\mathbf{k}-\mathbf{k}',\,\omega-\omega'}\,d\mathbf{k}'\,d\omega' \tag{II. 5}$$

By a similar method we could determine the higher order corrections to the amplitudes. However, we are not so much interested in the correction to the amplitude as in the effect of the wave interaction on the characteristic frequencies of the oscillations, and more precisely on their imaginary part, i.e. the damping. To determine the magnitude of the additional damping we multiply eqn. (II.1) by $C^*_{\mathbf{k}'\omega'}$ and then average the result with respect to the statistical ensemble, assuming, as before, that in zero approximation the

oscillations are uncorrelated (the random phase approximation). The right hand side, containing the product of three random variables, vanishes in zero approximation; the next approximation is to include successively in each of the three factors the first order correction $C^{(1)}$, which leads to

$$(\omega-\omega_{\mathbf{k}}-i\gamma_{\mathbf{k}})I_{\mathbf{k}\omega} = I_{\mathbf{k}\omega}\int V_{\mathbf{k}\omega,\mathbf{k}'\omega'}\frac{V_{\mathbf{k}''\omega'',\mathbf{k}\omega}+V_{\mathbf{k}''\omega'',-\mathbf{k}'-\omega'}}{\omega''-\omega_{\mathbf{k}''}+i\nu}I_{\mathbf{k}'\omega'}\,d\mathbf{k}'\,d\omega'+$$
$$+I_{\mathbf{k}\omega}\int V_{\mathbf{k}\omega,\mathbf{k}'\omega'}\frac{V_{\mathbf{k}'\omega',\mathbf{k}\omega}+V_{\mathbf{k}'\omega',-\mathbf{k}''-\omega''}}{\omega'-\omega_{\mathbf{k}'}+i\nu}I_{\mathbf{k}''\omega''}\,d\mathbf{k}'\,d\omega'+$$
$$+\frac{1}{\omega-\omega_{\mathbf{k}}-i\nu}\int V_{\mathbf{k}\omega,\mathbf{k}'\omega'}(V^{*}_{\mathbf{k}\omega,\mathbf{k}'\omega'}+V^{*}_{\mathbf{k}\omega,\mathbf{k}''\omega''})I_{\mathbf{k}'\omega'}I_{\mathbf{k}''\omega''}\,d\mathbf{k}'\,d\omega' \qquad \text{(II. 6)}$$

where $\mathbf{k}'' = \mathbf{k}-\mathbf{k}'$, $\omega'' = \omega-\omega'$.

The first two terms on the right hand side of this equation are proportional to $I_{\mathbf{k}\omega}$ and contribute therefore to altering the characteristic frequencies. For our purposes it is sufficient to consider only the imaginary part of these terms, since it is this part which describes the damping of the waves due to the non-linear interaction. Their real parts, giving the frequency, can be neglected on the strength of the condition $\gamma/\omega \ll 1$. Transferring the terms proportional to $I_{\mathbf{k}\omega}$ to the left, they can be combined together with $\gamma_{\mathbf{k}}I_{\mathbf{k}\omega}$ and we obtain the total growth rate $\tilde{\gamma}_{\mathbf{k}}$ (more precisely speaking the damping rate, since in the steady state the value of $\tilde{\gamma}_{\mathbf{k}}$ must be negative). Since the non-linear terms are small we can, within the framework of our approximation, replace the small imaginary part $\nu > 0$ by $-\tilde{\gamma}_{\mathbf{k}} > 0$ in the last term, and the relation (II.6) can then be rewritten in the following form:

$$\{(\omega-\omega_{\mathbf{k}})^2+\tilde{\gamma}_{\mathbf{k}}^2\}I_{\mathbf{k}\omega} = \tfrac{1}{2}\int |v_{\mathbf{k}\omega,\mathbf{k}'\omega'}|^2 I_{\mathbf{k}'\omega'}I_{\mathbf{k}''\omega''}\,d\mathbf{k}'\,d\omega' \qquad \text{(II. 7)}$$

where $v_{\mathbf{k}\omega,\mathbf{k}'\omega'} = V_{\mathbf{k}\omega,\mathbf{k}'\omega'}+V_{\mathbf{k}\omega,\mathbf{k}''\omega''}$ and the right hand side of (II.7) is now in a symmetrical form.

With a small $\tilde{\gamma}_{\mathbf{k}}$ the expression $[(\omega-\omega_{\mathbf{k}})^2+\tilde{\gamma}_{\mathbf{k}}^2]^{-1} \cong -\dfrac{\pi}{\tilde{\gamma}_{\mathbf{k}}}\delta(\omega-\omega_{\mathbf{k}})$, so that as previously $I_{\mathbf{k}\omega} = I_{\mathbf{k}}\delta(\omega-\omega_{\mathbf{k}})$. Substituting this last relation in (II.7) and considering that $\operatorname{Im}(\omega'-\omega_{\mathbf{k}'}+i\nu)^{-1} = -\pi\delta(\omega'-\omega_{\mathbf{k}'})$, we obtain the kinetic equation for sustained oscillations:

$$-\tilde{\gamma}_{\mathbf{k}}I_{\mathbf{k}} \equiv -\gamma_{\mathbf{k}}I_{\mathbf{k}}+\pi I_{\mathbf{k}}\operatorname{Re}\int v_{\mathbf{k}\omega,\mathbf{k}''\omega''}v_{\mathbf{k}''\omega'',\mathbf{k}\omega}\,\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\omega_{\mathbf{k}''})I_{\mathbf{k}'}\,d\mathbf{k}'-$$
$$-I_{\mathbf{k}}\int\frac{\operatorname{Im}(v_{\mathbf{k}\omega,\mathbf{k}''\omega''}v_{\mathbf{k}''\omega'',\mathbf{k}\omega})}{\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\omega_{\mathbf{k}''}}I_{\mathbf{k}'}\,d\mathbf{k}'$$
$$=\frac{\pi}{2}\int |v_{\mathbf{k}\omega,\mathbf{k}'\omega'}|^2 I_{\mathbf{k}'}I_{\mathbf{k}''}\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\omega_{\mathbf{k}''})\,d\mathbf{k}' \qquad \text{(II. 8)}$$

where the integral with respect to $\mathbf{k}'$ in the last term on the left hand side is the principal value, and $\mathbf{k}'' = \mathbf{k}-\mathbf{k}'$, $\omega'' = \omega-\omega'$.

In a transparent medium $v_{\mathbf{k}\omega,\mathbf{k}'\omega'}$ is real. For simplicity we shall only

consider this case here, leaving the general case to Section (2) where we shall consider the thermal motion of the particles.

Note here that the addition to the right hand side of eqn. (II.1) of a cubic term of the form

$$\int V_{\mathbf{k}\omega,\,\mathbf{k}'\omega',\,\mathbf{k}''\omega''} C_{\mathbf{k}'\omega'} C_{\mathbf{k}''\omega''} C_{\mathbf{k}-\mathbf{k}'-\mathbf{k}'',\,\omega-\omega'-\omega''}\, d\mathbf{k}'\, d\mathbf{k}''\, d\omega'\, d\omega'' \qquad \text{(II. 9)}$$

would lead to the appearance on the right hand side of eqn. (II.6) of an additional term

$$\int U_{\mathbf{k}\omega,\,\mathbf{k}'\omega'} I_{\mathbf{k}\omega} I_{\mathbf{k}'\omega'}\, d\mathbf{k}'\, d\omega' \qquad \text{(II. 10)}$$

where

$$U_{\mathbf{k}\omega,\,\mathbf{k}'\omega'} = V_{\mathbf{k}\omega,\,\mathbf{k}'\omega',\,-\mathbf{k}'-\omega'} + V_{\mathbf{k}\omega,\,\mathbf{k}\omega,\,\mathbf{k}'\omega'} + V_{\mathbf{k}\omega,\,\mathbf{k}'\omega',\,\mathbf{k}\omega} \qquad \text{(II. 11)}$$

Equation (II.8) defines the spectrum of the oscillations in the steady state, where the growth of the waves due to the instability is exactly compensated by their damping due to the non-linear interaction. In the absence of equilibrium the oscillation amplitude will vary with time with a growth rate equal to the difference between the linear growth rate and the non-linear damping. The kinetic equation for the waves, including non-linear effects, can therefore be obtained from (I.43) simply by replacing γ_k by $\tilde{\gamma}_k$, to give

$$\frac{\partial I_{\mathbf{k}}}{\partial t} + \mathbf{U}_{\mathbf{k}} \frac{\partial I_{\mathbf{k}}}{\partial \mathbf{r}} - \frac{\partial \omega_{\mathbf{k}}}{\partial \mathbf{r}} \frac{\partial I_{\mathbf{k}}}{\partial \mathbf{k}} =$$

$$= 2\gamma_{\mathbf{k}} I_{\mathbf{k}} - \pi \int \{2 v_{\mathbf{k}\mathbf{k}'} v_{\mathbf{k}'\mathbf{k}} I_{\mathbf{k}} I_{\mathbf{k}''} - |v_{\mathbf{k}\mathbf{k}'}|^2 I_{\mathbf{k}'} I_{\mathbf{k}''}\} \delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}''})\, d\mathbf{k}' \qquad \text{(II. 12)}$$

This equation, together with the equation for the averaged function, constitutes the basis for describing weakly turbulent systems. In the case of a homogeneous medium, where $\frac{\partial \omega_{\mathbf{k}}}{\partial \mathbf{r}} = 0$, this equation is analogous to the well known kinetic equation for phonons, the sound quanta in a solid (see ref. 11). The kinetic equation for waves has only very recently been applied to a plasma (refs. 10 and 13).

In this wave eqn. (II.12), only " three-wave " processes are considered; these are the decay of the wave **k** into **k**′, **k**″ and the inverse process of the merging of the two waves **k**′, **k**″ into one. The δ-function of the frequency difference appearing in (II.12) requires $\omega_{\mathbf{k}} = \omega_{\mathbf{k}}' + \omega_{\mathbf{k}}''$, and this condition, which can be shown to represent the conservation of energy, considerably limits the permissible region of interaction in **k**-space, and indeed for many forms for the dispersion relation $\omega = \omega(\mathbf{k})$ three-wave processes are completely forbidden. It is natural therefore to divide the possible dispersion relations into two groups, decay and non-decay relations, according to whether decay of one wave into two is or is not permitted.

In an isotropic medium, for instance, a relation of the form 1 (Fig. 10) where the phase velocity decreases with k, is a non-decay relation, while for a spectrum of type 2, where the phase velocity increases with k, the decay conditions can be satisfied.

For a non-decay dispersion relation, we must go to the next approximation to obtain a non-vanishing interaction, and we then consider "four-wave" processes in which the scattering of two waves on one another gives two further waves. In the kinetic wave equation such processes lead to terms which are cubic in the spectral function I. But as we shall show below,

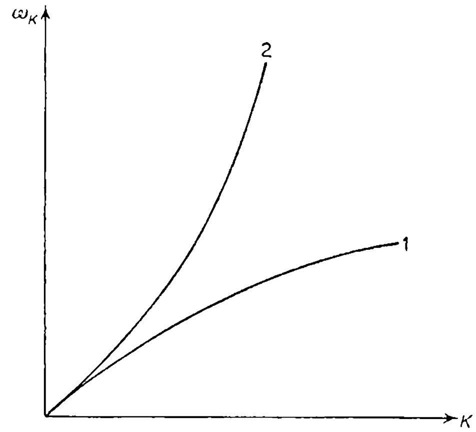

FIG. 10. Dispersion relations of decay (2) and non-decay (1) types

when the thermal motion of the particles is taken into account, additional terms quadratic in I appear, and these will in general dominate the cubic terms. We shall not, therefore, consider non-decay dispersion relations in any greater detail here.

Let us now discuss the two simplest examples of decay interactions in a plasma.

(b) *Interaction between Langmuir Waves and Ion-sound Waves*

Let us consider the simplest possible case, the longitudinal oscillations of a plasma in the absence of a magnetic field. We shall assume that the ion temperature T_i is considerably smaller than the electron temperature T_e, so that we can put approximately $T_i = 0$. Under these conditions both Langmuir and ion acoustic waves can propagate in the plasma. Each of these oscillations taken separately has a dispersion relation of the non-decay type. The dispersion relation for ion acoustic waves is of the form of curve 1 in Fig. 10, and the frequency of Langmuir oscillations approximates to ω_0, so that the sum of three frequencies cannot vanish. Therefore the only possible three-wave processes are those in which Langmuir waves are scattered by ion-acoustic waves, and we shall consider these in detail.

In the hydrodynamic approximation the oscillations are described by continuity equations

$$\frac{\partial n_j}{\partial t} + \operatorname{div}(n_j \mathbf{v}_j) = 0 \qquad \text{(II. 13)}$$

for the electron density n_e and the ion density n_i, the equations of motion for

each of the components

$$m_j n_j \frac{d\mathbf{v}_j}{dt} + \nabla(n_j T_j) = -\frac{e_j}{m_j} n_j \nabla\varphi \tag{II. 14}$$

and the equation for the electric potential

$$\Delta\varphi = -4\pi e(n_i - n_e). \tag{II. 15}$$

We shall use the index s for the ion-sound oscillations, the index l for the Langmuir oscillations. Let us consider first the Langmuir oscillations. In the case of the Langmuir oscillations the ions can be considered stationary, i.e. $n_i^l = 0$, so that eqn. (II.15) takes the form

$$\Delta\varphi^l = 4\pi e n_e^l \tag{II. 16}$$

Further, since decays within the Langmuir oscillation branch are prohibited, the non-linear term in the electron continuity equation and the non-linear term quadratic in velocity in the equation of motion for the electrons can be omitted. The non-linear coupling between the Langmuir and acoustic oscillations is given by a term on the right hand side of the electron equation of motion (II.14). Transforming to the Fourier representation we express n_e in terms of φ from (II.16). Taking the divergence of the electron equation of motion and expressing div $\mathbf{v}_e$ in terms of n_e (II.13), we obtain the equation for the potential of the Langmuir oscillations

$$(\omega^2 - \omega_e^2)\varphi_{\mathbf{k}\omega}^l = \int \frac{\mathbf{k}\mathbf{k}'}{k^2} \left\{ \frac{\omega_0^2}{n} \varphi_{\mathbf{k}'\omega'}^l n_{e\mathbf{k}-\mathbf{k}',\omega-\omega'}^s - \right.$$
$$\left. - \frac{e(\mathbf{k}-\mathbf{k}')^2}{m_e} \varphi_{\mathbf{k}'\omega'}^s \varphi_{\mathbf{k}-\mathbf{k}',\omega-\omega'}^l \right\} d\mathbf{k}'\, d\omega' \tag{II. 17}$$

where ω_e^2 is the square of the frequency of the Langmuir oscillations given by $\omega_e^2 = \omega_0^2 + \frac{3T_e}{m_e}k^2$. For k, k' small compared with the inverse Debye length the second term in the curly brackets in (II.17) is negligible, and recalling that $n_e^s = \frac{n_e}{T_e}\varphi^s$ we finally obtain

$$(\omega^2 - \omega_e^2)\varphi_{\mathbf{k}\omega}^l = \frac{e\omega_0^2}{T_e} \int \frac{\mathbf{k}\mathbf{k}'}{k^2} \varphi_{\mathbf{k}'\omega'}^l \varphi_{\mathbf{k}''\omega''}^s \, d\mathbf{k}'\, d\omega' \tag{II. 18}$$

where $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$, $\omega = \omega - \omega'$.

For the acoustic oscillations, the ion continuity equation and the equation of motion can be linearised, since decays within the ion branch are again prohibited. Expressing n_i in terms of φ^s from the linearised equations, we obtain from (II.15)

$$\left(1 - \frac{\Omega_0^2}{\omega^2}\right) k^2 \varphi_{\mathbf{k}\omega}^s = -4\pi e n_{e\mathbf{k}\omega}^s \tag{II. 19}$$

where $\Omega_0^2 = \frac{4\pi e^2 n}{m_i}$ is the square of the ion Langmuir frequency. At small k

the frequency of the ion-sound oscillations is considerably less than Ω_0, so that in (II.19) the unit in the brackets can be neglected.

To obtain the coupling between the ion-acoustic and Langmuir oscillations, we must take into account non-linear electron terms which would give low frequency beats. Taking the divergence of the electron equation of motion and noting that the term in $\nabla\varphi$ can now be linearised, we obtain in Fourier representation

$$nm_e \int (\mathbf{k}\mathbf{v}^l_{\mathbf{k}'\omega'})\left(\mathbf{k}'\mathbf{v}^l_{\mathbf{k}''\omega''} - \frac{\omega'}{n} n^l_{\mathbf{k}''\omega''}\right) d\mathbf{k}'\, d\omega' + k^2 T_e n^s_{e\mathbf{k}\omega} = k^2 ne\varphi^s_{\mathbf{k}\omega} \qquad \text{(II. 20)}$$

Since the frequency of the ion oscillations ω_s is small, we have approximately $\omega' n^l_{\mathbf{k}''\omega''} \approx -\omega'' n^l_{\mathbf{k}''\omega''} = -n\mathbf{k}''\mathbf{v}^l_{\mathbf{k}''\omega''}$. Neglecting the thermal corrections we can write $\mathbf{v}^l_{\mathbf{k}\omega} = -\dfrac{e\mathbf{k}}{m_e\omega}\varphi_{\mathbf{k}\omega}$ and then expressing $\mathbf{v}^l$ in terms of φ^l and n^s in terms of φ^s we obtain

$$(\omega^2 - \omega_s^2)\varphi^s_{\mathbf{k}\omega} = -\frac{e\omega^2}{m_e\omega_0^2}\int \frac{(\mathbf{k}\mathbf{k}')(\mathbf{k}\mathbf{k}'')}{k^2}\varphi^l_{\mathbf{k}'\omega'}\varphi^l_{\mathbf{k}''\omega''}\, d\mathbf{k}'\, d\omega' \qquad \text{(II. 21)}$$

where $\omega_s^2 = c_s^2 k^2 = \dfrac{T_e}{m_i}k^2$. This equation describes the interaction between the Langmuir and ion-acoustic oscillations.

In the linear approximation, to each wave vector $\mathbf{k}$ correspond two fast waves, $\omega_{\mathbf{k}} = \pm\omega_l$, and two slow waves, $\omega_{\mathbf{k}} = \pm\omega_s$. The spectral function of the potential φ for these oscillations will be denoted by $I^{\pm l}_{\mathbf{k}}$ and $\mathbf{I}^{\pm s}_{\mathbf{k}}$ respectively so that, for instance,

$$I^s_{\mathbf{k}\omega} = I^{+s}_{\mathbf{k}}\delta(\omega - kc_s) + I^{-s}_{\mathbf{k}}\delta(\omega + kc_s) \qquad \text{(II. 22)}$$

In addition, since $\mathbf{I}_{\mathbf{k}\omega} = \mathbf{I}_{-\mathbf{k},-\omega}$, we shall assume that $\omega_{-\mathbf{k}} = -\omega_{\mathbf{k}}$ and consequently the transformation $\omega, \mathbf{k} \to -\omega, -\mathbf{k}$ changes neither $\mathbf{I}_{\mathbf{k}}$ nor $\delta(\omega - \omega_{\mathbf{k}})$.

In principle it is possible to choose another method of differentiating between the two waves with the same wave vector, but propagating in opposite directions. Often, for instance, the frequency is defined to be positive and then the direction $\mathbf{k}$ defines the direction of propagation of the wave. In this system, however, a more complicated notation for the collision term between the waves is required. The transition from one representation to the other is simple and we shall use only the representation introduced above.

Near each of the characteristic frequencies, the difference $\omega^2 - \omega_{\mathbf{k}}^2$ can be approximated by $2\omega_{\mathbf{k}}(\omega - \omega_{\mathbf{k}})$, and the eqns. (II.18) and (II.21) assume the form of the model eqn. (II.1). The kinetic equations can therefore be written down by analogy; we obtain the following system for the longitudinal oscillations of a homogeneous plasma

$$\frac{\partial I^l_{\mathbf{k}}}{\partial t} + \mathbf{U}^l_{\mathbf{k}}\frac{\partial I^l_{\mathbf{k}}}{\partial \mathbf{r}} = \frac{\pi}{4}\frac{e^2\omega_0^2}{T_e^2}\int \cos^2\alpha (I^l_{\mathbf{k}} I^s_{\mathbf{k}''} - I^l_{\mathbf{k}'} I^s_{\mathbf{k}''})\delta(\omega^l_{\mathbf{k}} - \omega^l_{\mathbf{k}'} - \omega^s_{\mathbf{k}''})\, d\mathbf{k}' \qquad \text{(II. 23)}$$

$$\frac{\partial I_{\mathbf{k}}^{s}}{\partial t}+\mathbf{U}_{\mathbf{k}}^{s}\frac{\partial I_{\mathbf{k}}^{s}}{\partial \mathbf{r}}=\frac{\pi}{4}\frac{e^{2}\omega_{s}^{2}}{m_{e}^{2}\omega_{0}^{4}}\int k'^{2}(k-k'\cos\alpha)^{2}\cos^{2}\alpha I_{\mathbf{k}'}^{l}I_{\mathbf{k}''}^{l}\times$$
$$\times\,\delta(\omega_{\mathbf{k}}^{s}-\omega_{\mathbf{k}'}^{l}-\omega_{\mathbf{k}''}^{l})\,d\mathbf{k}'-\frac{\pi}{2}\frac{e^{2}\omega_{s}^{s}}{m_{e}T_{e}\omega_{0}}\int k^{2}\cos\alpha(k-k'\cos\alpha)\times$$
$$\times\,(k'-k\cos\alpha)I_{\mathbf{k}}^{s}I_{\mathbf{k}''}^{l}\delta(\omega_{\mathbf{k}}^{s}-\omega_{\mathbf{k}'}^{l}-\omega_{\mathbf{k}''}^{l})\,d\mathbf{k}' \qquad \text{(II. 24)}$$

where

$$\cos\alpha=\frac{\mathbf{k}\mathbf{k}'}{kk'},\qquad \mathbf{U}_{\mathbf{k}}^{\pm l}=\pm\frac{3T_{e}}{m_{e}\omega_{0}}\mathbf{k},\qquad \mathbf{U}_{\mathbf{k}}^{\pm s}=\pm\, c_{s}\frac{\mathbf{k}}{k}$$

It is evident from these equations that the main part in the interaction between the waves is played by the electron oscillations: beats between these oscillations excite ion-acoustic waves which then scatter the electron oscillations strongly. Let us for instance consider the case where a single wave with wave vector k_0 propagates in the initial state in the plasma, so that $\varphi^{l}=\varphi_{0}\delta(\omega-\omega_{k_0}^{l})\delta(k-k_{0})$ (Oraevskiĭ and Sagdeev (47)). Such a wave may decay into a Langmuir wave with $k'\approx -k_0$, propagating in the opposite direction and having a frequency $\omega_{k'}^{l}$ close to $\omega_{k_0}^{l}$, and an ion-sound wave with wave vector $k''\approx 2k_0$. In this case the condition $\omega_{k}^{l}-\omega_{k'}^{l}-\omega_{k''}^{s}=0$ can be satisfied.

To investigate this process it is convenient to revert to the eqns. (II.18), (II.21) for the amplitudes. For each of the excited waves we write $\omega=\omega_{\mathbf{k}}+i\gamma$ and linearise the equations for the amplitude relative to the perturbations $\varphi^{l}\delta(k+k_{0})\delta(\omega-\omega_{k}^{l})$ and $\varphi^{s}\delta(k-2k_{0})\delta(\omega-\omega_{k}^{s})$, we obtain

$$2i\gamma\omega^{l}\varphi^{l}=-\omega_{0}^{2}\frac{e\varphi_{0}}{T_{e}}\varphi^{s} \qquad \text{(II. 25)}$$

$$2i\gamma\omega_{s}\varphi^{s}=\frac{e\omega_{s}^{2}}{m_{e}\omega_{0}^{2}}k_{0}^{2}\varphi_{0}\varphi^{l} \qquad \text{(II. 26)}$$

whence, assuming $\omega^{l}\approx\omega_{0}$ we obtain

$$\gamma^{2}=\frac{1}{4}\frac{T_{e}k_{0}^{2}}{m_{e}}\frac{\omega_{s}}{\omega_{0}}\left(\frac{e\varphi_{0}}{T_{e}}\right)^{2} \qquad \text{(II. 27)}$$

Thus γ can be >0 and for a single Langmuir wave a " decay instability " occurs, in which the amplitude of a small perturbation representing a superposition of Langmuir and ion-acoustic waves, related to the main wave by the decay conditions, increases exponentially with time.

We could describe this process with the aid of the kinetic equations for the intensities I^{l} and I^{s}. However, in this case a difficulty arises because the substitution in the kinetic equation of functions of the form $I_{0}\delta(k-k_{0})$, $I^{l}\delta(k+k_{0})$ and $I^{s}\delta(k+2k_{0})$, leads to a divergence of the quadratic term. But if we recall that the function $\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\omega_{\mathbf{k}''})$ was in origin an approximation to a function of the form $\frac{\gamma}{\pi}[(\omega-\omega_{\mathbf{k}})^{2}+\gamma^{2}]^{-1}$, we see that this function must be replaced at the resonance point by $1/\pi\gamma$ and we again obtain (II.27).

This example shows again that the kinetic equation for the waves is strictly correct only for wave packets which are sufficiently broad in wave number space, with a smoothly varying function $I_{\mathbf{k}}$.

According to the kinetic eqns. (II.23), (II.24), the Langmuir oscillations alone cannot interact with one another and they must first excite ion oscillations. This process is a relatively slow one. But as soon as the amplitude I^s increases appreciably, the main process is the rapid scattering of the Langmuir waves at the ion inhomogeneities. Since the interaction between the Langmuir waves depends explicitly on the electron temperature, we should expect that an important part in this process may be played by processes involving the thermal motion of the electrons, which will be considered in Section 2.

(c) *Interaction between the Alfvén Waves and Magneto-acoustic Waves*

As a second example, we consider the oscillations of an ideally-conducting plasma in a homogeneous magnetic field (49). For simplicity we assume that the plasma pressure is much less than the magnetic pressure, so that in the equation of motion the plasma pressure can be neglected. Then the plasma oscillations will be described by the following system of magneto-hydrodynamic equations

$$m_i n \frac{d\mathbf{v}}{dt} = \frac{1}{4\pi}[\operatorname{curl}\mathbf{H}, \mathbf{H}] \tag{II. 28}$$

$$\frac{\partial \mathbf{H}}{\partial t} = \operatorname{curl}[\mathbf{v}\mathbf{H}] \tag{II. 29}$$

$$\frac{\partial n}{\partial t} = -\operatorname{div} n\mathbf{v} \tag{II. 30}$$

Suppose that in the stationary state the plasma is quiescent ($\mathbf{v} = 0$) and its density is constant. Linearising and transforming to a Fourier representation the first two equations become

$$-m_i n\omega \mathbf{v}_{\mathbf{k}\omega} = \frac{1}{4\pi}[[\mathbf{k}\mathbf{H}_{\mathbf{k}\omega}]\mathbf{H}] \tag{II. 31}$$

$$-\omega \mathbf{H}_{\mathbf{k}\omega} = [\mathbf{k}[\mathbf{v}_{\mathbf{k}\omega}\mathbf{H}]] \tag{II. 32}$$

The component of the velocity parallel to the mean magnetic field $\mathbf{H}$ has disappeared from these equations, so that it is sufficient to consider only oscillations transverse to the field. Expressing $\mathbf{H}_{\mathbf{k}\omega}$ in terms of $\mathbf{V}_{\mathbf{k}\omega}$ with the aid of (II.32) and substituting the result in (II.31), we obtain the equation for the velocity

$$\omega^2 \mathbf{v}_{\mathbf{k}\omega} = c_A^2[k_z^2 \mathbf{v}_{\mathbf{k}\omega} + \mathbf{k}(\mathbf{k}\mathbf{v}_{\mathbf{k}\omega})] \tag{II. 33}$$

where $c_A^2 = \dfrac{H^2}{4\pi m_i n}$ is the square of the Alfvén velocity and k_z the component of the wave vector parallel to the unperturbed magnetic field $\mathbf{H}$. From this

we obtain for the velocity component $\mathbf{v}^s_{\mathbf{k}\omega}$ in the plane (**kH**),

$$\omega^2 = \omega_s^2 = c_A^2 k^2 \tag{II. 34}$$

and for the component $v^a_{\mathbf{k}\omega}$, perpendicular to both **k** and **H** we have

$$\omega^2 = \omega_a^2 = c_A^2 k_z^2 \tag{II. 35}$$

Thus, for a low pressure plasma, i.e. $\beta = 8\pi p/H^2 \ll 1$, two types of waves can propagate, namely Alfvén waves with a frequency ω_a, and magneto-acoustic waves with a frequency ω_s. In addition stationary perturbations of the density ($\omega = 0$) are possible; these correspond to the slow magneto-acoustic waves which are obtained when the plasma pressure is not neglected, but these will not be considered here.

The phase velocity of the magnetosonic waves is a constant c_A. It follows, therefore, that the non-linear interaction is very important for waves propagating in the same direction. This interaction leads in particular to the steepening of the wave fronts of an initially sinusoidal wave by the generation of higher harmonics. But for waves propagating in different directions, the dispersion relation does not permit decay processes.

The Alfvén oscillations propagate only parallel to the magnetic field H, and this with the same phase velocity c_A. We might expect, therefore, that these oscillations would interact strongly with one another. However, the matrix element of the interaction of two Alfvén waves propagating in the same direction can be shown to vanish, and as a result a single Alfvén wave can propagate with any finite amplitude.

Thus, neither the Alfvén nor the magneto-acoustic waves by themselves are decay waves. However, each of them may decay into a pair of Alfvén and magneto-acoustic waves. The matrix elements of this interaction can be obtained from the equations of motion (II.28)–(II.30). Because of their cumbersomeness we shall not quote the general form of the expressions (these can be found in ref. (49)), but shall confine ourselves to the limiting case of almost transverse propagation where $k_z \ll k$. Neglecting in the matrix elements the quantity k_z compared with k, it is easy to transform the equations for v^s and v^a into the following form

$$(\omega^2 - k^2 c_A^2) v^s_{\mathbf{k}\omega} = \int \frac{kk'}{k''} \{(v^s_{\mathbf{k}'\omega'} v^a_{\mathbf{k}''\omega''} - v^a_{\mathbf{k}'\omega'} v^s_{\mathbf{k}''\omega''}) \sin\alpha\cos\alpha - \\ - v^a_{\mathbf{k}'\omega'} v^a_{\mathbf{k}''\omega''} \sin^2\alpha\}\, d\mathbf{k}'\, d\omega' \tag{II. 36}$$

$$(\omega^2 - k_z^2 c_A^2) v^a_{\mathbf{k}\omega} = \int \frac{kk'}{k''} (\sin\alpha\cos\alpha\, v^s_{\mathbf{k}'\omega'} v^s_{\mathbf{k}''\omega''} + v^s_{\mathbf{k}'\omega'} v^a_{\mathbf{k}''\omega''} \sin^2\alpha + \\ + v^a_{\mathbf{k}'\omega'} v^s_{\mathbf{k}''\omega''} \cos^2\alpha)\, d\mathbf{k}'\, d\omega' + \int \frac{kk'}{k''} \sin\alpha\cos\alpha \times \\ \times \left(\omega + c_A^2 \frac{\omega k_z'^2 - \omega' k_z^2}{\omega'\omega''}\right) v^a_{\mathbf{k}'\omega'} v^a_{\mathbf{k}''\omega''}\, d\mathbf{k}'\, d\omega' \tag{II. 37}$$

where $\sin\alpha = \dfrac{[\mathbf{kk}']_z}{kk'}$.

In the equation for v^a we have deliberately conserved the second integral term although in fact it vanishes for oscillations of such low intensity that the frequency shifts are negligible, so that we can write

$$\omega'^2 = c_A^2 k_z'^2, \ \omega^2 = c_A^2 k_z^2$$

In actual fact, at very small k_z and k_z', and a sufficiently large amplitude of the oscillations, we may find that the spectral function $|v^a_{\mathbf{k}\omega}|^2$ is broadened to cover a frequency range which may be comparable to or even larger than $k_z c_A$. In this case we can no longer neglect the Alfvén–Alfvén interaction because a transition takes place to strong turbulence where the restriction on " three-wave " processes due to the δ-function of frequency disappears.

We shall not here write out the kinetic equations for the Alfvén and magneto-acoustic waves, since they can be obtained without difficulty by analogy with the case of the Langmuir and ion acoustic waves considered above. It is sufficient to note that according to eqns. (II.36) and (II.37), the matrix element of the interaction is of the order of unity for v^a, $v^s \sim c_A$. It follows, therefore, that the lifetime τ for decay in the case of a single wave with amplitude v_0 is determined by the quantity $c_A^2/\omega_{a,\,s} v_0$, and the characteristic energy exchange time between different modes in a diffuse wave packet is of the order of $c_A^2/\omega_{a,\,s} v_0^2$.

2. Interaction of Waves in a Plasma with Consideration of the Thermal Motion of the Particles

In the derivation of eqn. (II.8) we did not take into account the specific properties of a plasma, i.e. a system of charged particles interacting with an electromagnetic field, as the medium supporting the oscillations. However, as we know from linear theory, the discreteness of the medium gives rise to a specific damping of the waves, namely Landau damping, related to the resonant interaction of the waves with the particles. We should naturally expect that some corresponding effect should appear in the non-linear case.

As we have seen above, the principal effect of the non-linearity is to lead to the appearance of beat oscillations with combination frequencies $\omega - \omega'$ and wave vectors $\mathbf{k} - \mathbf{k}'$. The resonant interaction between the particles and these beats gives rise to an additional damping of the waves which we shall call non-linear Landau damping. In the terminology of waves and particles this process corresponds to the scattering of waves by particles, i.e. in absorption of the wave k and re-emission of the wave k', whereas the linear Landau damping corresponds simply to an absorption of the wave. This non-linear damping effect is of the same order of magnitude as the decay processes, and when the dispersion relation is of the non-decay type it is the most important non-linear effect. To consider the non-linear interaction of the waves with the particles we must set up a kinetic equation for the waves on the basis of the non-linear kinetic equation for the particles. We then obtain simultaneously the effect of the waves on the averaged distribution function.

(a) *Kinetic Wave Equation with consideration of the Thermal Motion of the Particles*

In order to keep the discussion as simple as possible we consider, to start with, the case of longitudinal Langmuir oscillations, and we then show how the equations can be extended to the general case. Suppose that in the equilibrium state the plasma is homogeneous and there are no electric and magnetic fields. We divide the electron distribution function into two parts, one averaged with respect to time f and one oscillating part f' which we expand as a Fourier integral. Separating the kinetic equation into two by taking suitable averages, we obtain

$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla f = -i\frac{e}{m}\frac{\partial}{\partial \mathbf{v}}\int \mathbf{k}\langle \varphi^*_{\mathbf{k}'\omega'} f_{\mathbf{k}\omega}\rangle\, d\mathbf{k}\, d\omega\, d\mathbf{k}'d\omega' \tag{II. 38}$$

$$-i(\omega-\mathbf{k}\mathbf{v})f_{\mathbf{k}\omega} = -\frac{e}{m}i\mathbf{k}\frac{\partial f}{\partial \mathbf{v}}\varphi_{\mathbf{k}\omega} -$$
$$-i\frac{e}{m}\frac{\partial}{\partial \mathbf{v}}\int \mathbf{k}'(\varphi_{\mathbf{k}'\omega'} f_{\mathbf{k}-\mathbf{k}',\,\omega-\omega'} - \langle\varphi_{\mathbf{k}'\omega'} f_{\mathbf{k}-\mathbf{k}',\,\omega-\omega'}\rangle)\, d\mathbf{k}'d\omega \tag{II. 39}$$

where $\varphi_{\mathbf{k}\omega}$ is the Fourier component of the potential of the electric field, given by

$$\varphi_{\mathbf{k}\omega} = -\frac{4\pi e}{k^2}\int f_{\mathbf{k}\omega}\, d\mathbf{v} \tag{II. 40}$$

Neglecting the quadratic terms in the eqn. (II.39), we can express $f_{\mathbf{k}\omega}$ linearly in terms of $\varphi_{\mathbf{k}\omega}$:

$$f_{\mathbf{k}\omega} = \frac{e}{m}(\omega-\mathbf{k}\mathbf{v}+i\nu)^{-1}\mathbf{k}\frac{\partial f}{\partial \mathbf{v}}\varphi_{\mathbf{k}\omega} \tag{II. 41}$$

where we have introduced the small positive quantity $\nu \to 0$, allowing for a small damping, in order to pass round the pole correctly. Substituting this expression into (II.40), we obtain

$$\varepsilon(\mathbf{k}, \omega)\varphi_{\mathbf{k}\omega} = 0 \tag{II. 42}$$

which gives the dispersion relation $\varepsilon(\mathbf{k}, \omega) = 0$. The quantity

$$\varepsilon(\mathbf{k}, \omega) = 1 + \frac{4\pi e^2}{mk^2}\int \frac{\mathbf{k}\dfrac{\partial f}{\partial \mathbf{v}}}{(\omega-\mathbf{k}\mathbf{v}+i\nu)}\partial \mathbf{v} \tag{II. 43}$$

represents the dielectric permeability of the plasma.

Now let us consider the solution of the non-linear equation assuming that the amplitude of the oscillations is small. We write eqn. (II.39) in the more compact form

$$f_{\mathbf{k}\omega} = (\mathbf{g}_{\mathbf{k}\omega}\mathbf{k})f\varphi_{\mathbf{k}\omega} + \int(\mathbf{g}_{\mathbf{k}\omega}\mathbf{k}')(\varphi_{\mathbf{k}'\omega'} f_{\mathbf{k}-\mathbf{k}',\,\omega-\omega'} - \langle\varphi_{\mathbf{k}'\omega'} f_{\mathbf{k}-\mathbf{k}',\,\omega-\omega'}\rangle)\, d\mathbf{k}'\, d\omega' \tag{II. 44}$$

where g represents the following operator

$$\mathbf{g} = (\omega-\mathbf{k}\mathbf{v}+i\nu)^{-1}\frac{e}{m}\frac{\partial}{\partial \mathbf{v}} \tag{II. 45}$$

In eqn. (II.44) the quadratic terms can be considered small. Making use of this, we integrate this equation and represent $f_{\mathbf{k}\omega}$ in the following form

$$f_{\mathbf{k}\omega} = (\mathbf{g}_{\mathbf{k}\omega}\mathbf{k})f\varphi_{\mathbf{k}\omega} + \int (\mathbf{g}_{\mathbf{k}\omega}\mathbf{k}')(\mathbf{g}_{\mathbf{k}-\mathbf{k}'}(\mathbf{k}-\mathbf{k}'))f\{\varphi_{\mathbf{k}'\omega'}\varphi_{\mathbf{k}-\mathbf{k}',\omega-\omega'} - \\ - \langle\varphi_{\mathbf{k}'\omega'}\varphi_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\rangle\}\, d\mathbf{k}'\, d\omega' + \int (\mathbf{g}_{\mathbf{k}\omega}\mathbf{k}')(\mathbf{g}_{\mathbf{k}-\mathbf{k}'}\mathbf{k}'')\times \\ \times(\mathbf{g}_{\mathbf{k}-\mathbf{k}'-\mathbf{k}''}(\mathbf{k}-\mathbf{k}'-\mathbf{k}''))f\{\varphi_{\mathbf{k}'\omega'}\varphi_{\mathbf{k}''\omega''}\varphi_{\mathbf{k}-\mathbf{k}'-\mathbf{k}'',\omega-\omega'-\omega''} - \\ - \varphi_{\mathbf{k}'\omega'}\langle\varphi_{\mathbf{k}''\omega''}\varphi_{\mathbf{k}-\mathbf{k}'-\mathbf{k}'',\omega-\omega'-\omega''}\rangle - \langle\varphi_{\mathbf{k}'\omega'}\varphi_{\mathbf{k}''\omega''}\varphi_{\mathbf{k}-\mathbf{k}'-\mathbf{k}'',\omega-\omega'-\omega''}\rangle\}\times \\ \times d\mathbf{k}'\, d\omega'\, d\mathbf{k}''\, d\omega'' + \dots \qquad \text{(II. 46)}$$

Within our approximation we can neglect the terms of the fourth and higher orders in the oscillation amplitude in (II.46). Substituting this equation into (II.40) we obtain the following non-linear equation for the potential $\varphi_{\mathbf{k}\omega}$:

$$\varepsilon(\mathbf{k},\omega)\varphi_{\mathbf{k}\omega} = \int V_{\mathbf{k}\omega,\mathbf{k}'\omega'}\{\varphi_{\mathbf{k}'\omega'}\varphi_{\mathbf{k}-\mathbf{k}',\omega-\omega'} - \langle\varphi_{\mathbf{k}'\omega'}\varphi_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\rangle\}\, d\mathbf{k}'\, d\omega' + \\ + \int V_{\mathbf{k}\omega,\mathbf{k}'\omega',\mathbf{k}''\omega''}\{\varphi_{\mathbf{k}'\omega'}\varphi_{\mathbf{k}''\omega''}\varphi_{\mathbf{k}-\mathbf{k}',\omega'-\omega'} - \\ - \varphi_{\mathbf{k}'\omega'}\langle\varphi_{\mathbf{k}''\omega''}\varphi_{\mathbf{k}-\mathbf{k}'-\mathbf{k}'',\omega-\omega'-\omega''}\rangle - \langle\varphi_{\mathbf{k}'\omega'}\varphi_{\mathbf{k}''\omega''}\varphi_{\mathbf{k}-\mathbf{k}'-\mathbf{k}'',\omega-\omega'-\omega''}\rangle\}\times \\ d\mathbf{k}'\, d\omega'\, d\mathbf{k}''\, d\omega'' \qquad \text{(II. 47)}$$

where the matrix elements are given by the following relations

$$V_{\mathbf{k}\omega,\mathbf{k}'\omega'} = -\frac{4\pi e}{k^2}\int (\mathbf{g}_{\mathbf{k}\omega}\mathbf{k}')(\mathbf{g}_{\mathbf{k}-\mathbf{k}',\omega-\omega'}(\mathbf{k}-\mathbf{k}'))f\, d\mathbf{v} \qquad \text{(II. 48)}$$

$$V_{\mathbf{k}\omega,\mathbf{k}'\omega',\mathbf{k}''\omega''} = -\frac{4\pi e}{k^2}\int (\mathbf{g}_{\mathbf{k}\omega}\mathbf{k}')(\mathbf{g}_{\mathbf{k}-\mathbf{k}'}\mathbf{k}'')(\mathbf{g}_{\mathbf{k}-\mathbf{k}'-\mathbf{k}''}(\mathbf{k}-\mathbf{k}'-\mathbf{k}''))\, d\mathbf{v} \qquad \text{(II. 49)}$$

Equation (II.47) is similar in structure to our model eqn. (II.1), supplemented by a cubic term of the form (II.9). We can therefore immediately use the results of the preceding section and, by analogy with (II.6) and (II.7), obtain the following kinetic equation for the waves with consideration of the thermal motion of the particles

$$\varepsilon(\mathbf{k},\omega)I_{\mathbf{k}\omega} = I_{\mathbf{k}\omega}\frac{4\pi e}{k^2}\int (\mathbf{k}'\mathbf{g}_{\mathbf{k}\omega})\{(\mathbf{k}\mathbf{g}_{\mathbf{k}''\omega''})(\mathbf{k}'\mathbf{g}_{-\mathbf{k}'-\omega'})f + \\ + (\mathbf{k}'\mathbf{g}_{\mathbf{k}''\omega''})(\mathbf{k}\mathbf{g}_{\mathbf{k}\omega})f\}I_{\mathbf{k}'\omega'}\, d\mathbf{v}\, d\mathbf{k}'\, d\omega' + \\ + I_{\mathbf{k}\omega}\int \frac{v_{\mathbf{k}\omega,\mathbf{k}''\omega''}v_{\mathbf{k}''\omega'',\mathbf{k}\omega}}{\varepsilon_+(\mathbf{k}'',\omega'')} I_{\mathbf{k}'\omega'}\, d\mathbf{k}'\, d\omega' + \\ + \frac{1}{2\varepsilon_+^*(\mathbf{k},\omega)}\int |v_{\mathbf{k}\omega,\mathbf{k}'\omega'}|^2 I_{\mathbf{k}'\omega'}I_{\mathbf{k}''\omega''}\, d\mathbf{k}'\, d\omega' \qquad \text{(II. 50)}$$

where

$$\mathbf{k}'' = \mathbf{k}-\mathbf{k}', \qquad \omega'' = \omega-\omega', \qquad \varepsilon_+(\mathbf{k},\omega) = \varepsilon(\mathbf{k},\omega+i\nu)$$

$$v_{\mathbf{k}\omega,\mathbf{k}'\omega'} = V_{\mathbf{k}\omega,\mathbf{k}'\omega'} + V_{\mathbf{k}\omega,\mathbf{k}''\omega''} = v_{\mathbf{k}\omega,\mathbf{k}''\omega''}$$

The quadratic terms on the right hand side of eqn. (II.50) describe the additional damping and the shift of the characteristic frequencies due to the

non-linear interaction between the waves. For $\gamma/\omega \ll 1$, i.e. $\varepsilon''/\varepsilon' \ll 1$ where $\varepsilon' = \mathrm{Re}\,\varepsilon$ and $\varepsilon'' = \mathrm{Im}\,\varepsilon$, eqn. (II.50) can be solved by the method of successive approximations. The zero approximation gives $\varepsilon' I_{\mathbf{k}\omega} = 0$ which has the solution $I_{\mathbf{k}\omega} = I_{\mathbf{k}}\,\delta(\omega - \omega_{\mathbf{k}})$, where $\omega_{\mathbf{k}}$ is the characteristic frequency. In the next approximation it is sufficient to consider only the imaginary part of eqn. (II.50). Denoting by $\gamma_{\mathbf{k}}$ the linear growth rate $\gamma_{\mathbf{k}} = -\,\varepsilon''\left(\dfrac{\partial \varepsilon'}{\partial \omega}\right)^{-1}$ and adding a term including the derivatives with respect to time, we write down the imaginary part of eqn. (II.50) in the following form

$$\begin{aligned}\frac{1}{2}\frac{\partial I_{\mathbf{k}}}{\partial t} = \gamma_{\mathbf{k}} I_{\mathbf{k}} + I_{\mathbf{k}}\left(\frac{\partial \varepsilon'}{\partial \omega}\right)^{-1}_{\mathbf{k}\omega_{\mathbf{k}}} \mathrm{Im}\,\frac{4\pi e}{\mathbf{k}^2}\int (\mathbf{k}'\mathbf{g}_{\mathbf{k}\omega})\{(\mathbf{k}\mathbf{g}_{\mathbf{k}''\omega''})(\mathbf{k}'\mathbf{g}_{-\mathbf{k}'-\omega'})f + \\ + (\mathbf{k}'\mathbf{g}_{\mathbf{k}''\omega''})(\mathbf{k}\mathbf{g}_{\mathbf{k}\omega})f\} I_{\mathbf{k}'}\, d\mathbf{v}\, d\mathbf{k}' + \\ + I_{\mathbf{k}}\left(\frac{\partial \varepsilon'}{\partial \omega}\right)^{-1}_{\mathbf{k}\omega_{\mathbf{k}}} \mathrm{Im}\int \frac{v_{\mathbf{k}\omega,\,\mathbf{k}''\omega''}\, v_{\mathbf{k}''\omega'',\,\mathbf{k}\omega}\, I_{\mathbf{k}'}}{\varepsilon(\mathbf{k}'', \omega_{\mathbf{k}} - \omega_{\mathbf{k}'} + i\nu)}\, d\mathbf{k}' + \\ + \frac{\pi}{2}\left(\frac{\partial \varepsilon'}{\partial \omega}\right)^{-2}_{\mathbf{k}\omega_{\mathbf{k}}} \int |v_{\mathbf{k}\omega,\,\mathbf{k}'\omega'}|^2 \delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}''}) I_{\mathbf{k}'} I_{\mathbf{k}''}\, d\mathbf{k}' \qquad \text{(II. 51)}\end{aligned}$$

The last term and the subtracted part of the penultimate term describe the wave decay processes and the remaining non-linear terms describe the scattering of the waves by particles; they include the linear Landau damping.

We have written down this equation for the electrons only. To include the ions it is sufficient to consider their contribution to ε and to the matrix elements $v_{\mathbf{k}\omega,\,\mathbf{k}'\omega'}$, and also to sum the second term in the right hand side of (II.51) over the two particle types. No difficulties are encountered in generalising these equations to the cases in which the plasma is located in a magnetic field: the only change is in the form of the operators $\mathbf{g}_{\mathbf{k}\omega}$. The generalisation to the case of randomly polarized oscillations gives considerably more complicated expressions, because the full dispersion function given by eqn. (I.33) must be used in place of ε and the one eqn. (II.51) must be replaced by separate equations for each of the possible polarizations.

(b) *Thermal Fluctuations*

So far we have throughout understood by f the distribution function averaged over small macroscopic volumes in phase space and therefore a continuous function. Similarly we have omitted from consideration the thermal fluctuations related to the discreteness of matter. We shall now drop this assumption and discuss the effect of thermal fluctuations.

The kinetic Vlasov equation without the collision term can be also written down for a microscopic distribution function

$$f_m(\mathbf{r}, \mathbf{v}, t) = \sum_j \delta(\mathbf{r} - \mathbf{r}_j)\delta(\mathbf{v} - \mathbf{v}_j)$$

where the summation is over all particles of the given type. Let us split up the function f_m into two parts: an averaged part f and a fluctuating part f^{μ}, which

vanishes when averaged over the macroscopic volume. Neglecting the interaction between particles, which is admissible for a rarified plasma for which the Debye number $N_D = nD^3 \gg 1$, we obtain the correlation coefficients

$$\langle f^{\mu}(\mathbf{r}, \mathbf{v}, t) f^{\mu}(\mathbf{r}', \mathbf{v}', t')\rangle = \delta(\mathbf{v}-\mathbf{v}')\delta(\mathbf{r}-\mathbf{r}'-\mathbf{v}(t-t'))f(\mathbf{v}) \qquad \text{(II. 52)}$$

in co-ordinate representation and

$$\langle f^{\mu}_{\mathbf{k}\omega}(\mathbf{v}) f^{\mu}_{\mathbf{k}'\omega'}(\mathbf{v}')\rangle = \delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega')\delta(\omega-\mathbf{kv})\frac{f(\mathbf{v})}{(2\pi)^3} \qquad \text{(II. 53)}$$

in the Fourier representation.

The correlation function (II.52) corresponds to freely moving particles and satisfies the kinetic equation with no interaction. In the presence of oscillations the macroscopic function $f(\mathbf{r}, \mathbf{v}, t)$ also becomes random, i.e. macroscopic fluctuations arise, and only these have been considered earlier. It is easily seen that the thermal fluctuations were omitted when we expressed $f_{\mathbf{k}\omega}$ in terms of $\varphi_{\mathbf{k}\omega}$. In fact the general solution of the linearised kinetic equation for the longitudinal oscillations has the form

$$f_{\mathbf{k}\omega} = \frac{e}{m}\mathbf{k}\frac{\partial f}{\partial \mathbf{v}}\frac{\varphi_{\mathbf{k}\omega}}{\omega-\mathbf{kv}+i\nu} + f^{\mu}_{\mathbf{k}\omega}(\mathbf{v}) \qquad \text{(II. 54)}$$

where the function $f^{\mu}_{\mathbf{k}\omega}$ satisfies the following equation

$$(\omega-\mathbf{kv})f^{\mu}_{\mathbf{k}\omega} = 0 \qquad \text{(II. 55)}$$

In the absence of external beams we can understand $f^{\mu}_{\mathbf{k}\omega}$ as a fluctuating function satisfying relation (II.52). Previously we neglected this additional term, assuming that the amplitude of turbulent oscillations considerably exceeds the thermal level. With the inclusion of the term $f^{\mu}_{\mathbf{k}\omega}$ the equation for $\varphi_{\mathbf{k}\omega}$ becomes

$$\varepsilon(\mathbf{k}\omega)\varphi_{\mathbf{k}\omega} = -\frac{4\pi e}{k^2}\int f^{\mu}_{\mathbf{k}\omega}\, d\mathbf{v} \qquad \text{(II. 56)}$$

that is, an additional noise source appears. Accordingly, in eqn. (II.50) additional terms appear taking into account the thermal fluctuations, and it assumes the form (see (213))

$$\begin{aligned}\varepsilon(\mathbf{k}\omega)I_{\mathbf{k}\omega} = {} & \frac{2}{\pi}\frac{e^2}{k^4\varepsilon_+^*(\mathbf{k}\omega)}\int f(\mathbf{v})\delta(\omega-\mathbf{kv})\,d\mathbf{v} + \\ & + \frac{4\pi e}{k^2}I_{\mathbf{k}\omega}\int\int (\mathbf{k}'\mathbf{g}_{\mathbf{k}\omega})\{(\mathbf{k}\mathbf{g}_{\mathbf{k}''\omega''})(\mathbf{k}'\mathbf{g}_{-\mathbf{k}'-\omega'})f + \\ & + (\mathbf{k}'\mathbf{g}_{\mathbf{k}''\omega''})(\mathbf{k}\mathbf{g}_{\mathbf{k}\omega})f\}\,d\mathbf{v}\,I_{\mathbf{k}'\omega'}\,d\mathbf{k}'\,d\omega' + \\ & + I_{\mathbf{k}\omega}\int\frac{v_{\mathbf{k}\omega,\mathbf{k}''\omega''}v_{\mathbf{k}''\omega'',\mathbf{k}\omega}}{\varepsilon_+(\mathbf{k}''\omega'')}I_{\mathbf{k}'\omega'}\,d\mathbf{k}'\,d\omega' + \\ & + \frac{1}{2\varepsilon_+^*(\mathbf{k}\omega)}\int |v_{\mathbf{k}\omega,\mathbf{k}'\omega'}|^2 I_{\mathbf{k}'\omega'}I_{\mathbf{k}''\omega''}\,d\mathbf{k}'\,d\omega' \end{aligned} \qquad \text{(II. 57)}$$

The first term on the right hand side of this equation represents the thermal fluctuations. We can represent eqn. (II.51) symbolically in the form

$$\frac{\partial I}{\partial t} = 2\gamma I + q - \alpha I^2 \tag{II. 58}$$

where the first term describes the build-up of the oscillations with growth rate γ, q represents the source due to thermal noise, and the non-linear term describes the interaction between the waves. Thus in the stationary turbulent state for a large growth rate, we may neglect q, and thus $I = \alpha/2\gamma$. On the other hand, when γ is negative and not very small, we can neglect the non-linear term in eqn. (II.58) so that $I = q/2|\gamma|$. In this case only thermal noise is present in the plasma. For $\gamma \to 0$ this noise level diverges and to determine I we must preserve the non-linear term. This is illustrated in Fig. 11 which shows the transition from thermal to turbulent fluctuations.

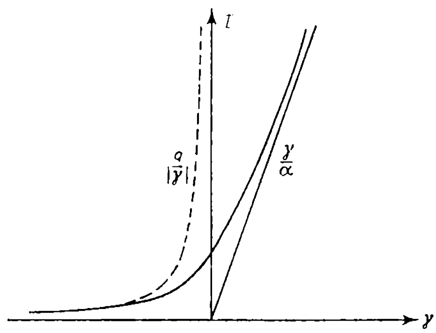

FIG. 11. Dependence of noise intensity on growth rate of small oscillations

(c) *Wave-particle Interaction*

Let us now return to the equation for the averaged function (II.38). The expression on the right hand side of this equation, which we shall again denote by S_{ef}, represents the collisions between the particles and waves and can be written in the form

$$S_{ef} = -\mathrm{Im}\,\frac{e}{m}\frac{\partial}{\partial \mathbf{v}}\int \mathbf{k}P_{\mathbf{k}\omega}(\mathbf{v})\,d\mathbf{k}\,d\omega \tag{II. 59}$$

where the correlation function $P_{\mathbf{k}\omega}$ is defined by

$$P_{\mathbf{k}\omega}(\mathbf{v})\delta(\omega-\omega')\delta(\mathbf{k}-\mathbf{k}') = \langle \varphi^*_{\mathbf{k}'\omega'} f_{\mathbf{k}\omega}\rangle \tag{II. 60}$$

Utilizing eqn. (II.46) for $f_{\mathbf{k}\omega}$ and adding to it the term $f^{\mu}_{\mathbf{k}\omega}$, related to the thermal fluctuations (see (II.53)), we obtain in the random phase approximation

$$P_{\mathbf{k}\omega}(\mathbf{v}) = -\frac{4\pi e f}{(2\pi)^3 k^2 \varepsilon^*_{+}(\mathbf{k},\omega)}\delta(\omega-\mathbf{k}\mathbf{v}) + I_{\mathbf{k}\omega}(\mathbf{k}\mathbf{g}_{\mathbf{k}\omega})f -$$
$$- I_{\mathbf{k}\omega}\int (\mathbf{k}'\mathbf{g}_{\mathbf{k}\omega})\{(\mathbf{k}\mathbf{g}_{\mathbf{k}''\omega''})(\mathbf{k}'\mathbf{g}_{-\mathbf{k}'-\omega'})f + (\mathbf{k}'\mathbf{g}_{\mathbf{k}''\omega''})(\mathbf{k}\mathbf{g}_{\mathbf{k}\omega})f\}I_{\mathbf{k}'\omega'}\,d\mathbf{k}'\,d\omega' +$$

$$+ I_{\mathbf{k}\omega} \int \frac{v_{\mathbf{k}''\omega'',\mathbf{k}\omega}}{\varepsilon_+(\mathbf{k}'',\omega'')} (\mathbf{k}'\mathbf{g}_{\mathbf{k}\omega})\{(\mathbf{k}''\mathbf{g}_{\mathbf{k}''\omega''})f + (\mathbf{k}'\mathbf{g}_{\mathbf{k}'\omega'})f\} I_{\mathbf{k}'\omega'}\, d\mathbf{k}'\, d\omega' +$$

$$+ \frac{1}{\varepsilon_+^*(\mathbf{k},\omega)} \int v^*_{\mathbf{k}\omega,\mathbf{k}'\omega'} (\mathbf{k}'\mathbf{g}_{\mathbf{k}\omega})(\mathbf{k}''\mathbf{g}_{\mathbf{k}''\omega''}) f I_{\mathbf{k}'\omega'} I_{\mathbf{k}''\omega''}\, d\mathbf{k}'\, d\omega' \quad \text{(II. 61)}$$

where $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$, $\omega'' = \omega - \omega'$, $\varepsilon_+(\mathbf{k}, \omega) = \varepsilon(\mathbf{k}, \omega + i\nu)$.

In this expression the first term describes the slowing down of the particles due to the polarization of the medium and the radiation of longitudinal waves by the Cerenkov effect. The second term, which is linear in $I_{\mathbf{k}\omega}$, has already been considered earlier when treating the quasi-linear approximation, and the remaining terms in (II.61) arise from the non-linear interaction.

Let us first consider the simple case of a stationary stable plasma where the quadratic terms in I in (II.57) and (II.61) can be neglected. Determining from the kinetic wave eqn. (II.57) the equilibrium intensity of the fluctuations $I_{\mathbf{k}\omega}$ and substituting the result into the second term in (II.61), we obtain

$$P_{\mathbf{k}\omega}(\mathbf{v}) = -\frac{4\pi e f \delta(\omega - \mathbf{k}\mathbf{v})}{(2\pi)^3 k^2 \varepsilon_+^*(\mathbf{k},\omega)} + \frac{e}{m} \frac{\mathbf{k}\dfrac{\partial f}{\partial \mathbf{v}}}{\omega - \mathbf{k}\mathbf{v} + i\nu} \cdot \frac{2e^2}{\pi|\varepsilon(\mathbf{k},\omega)|^2 k^4} \int f(\mathbf{v}')\delta(\omega - \mathbf{k}\mathbf{v}')\, d\mathbf{v}' \quad \text{(II. 62)}$$

Substituting this expression in (II.59) and remembering that according to (II.43)

$$\varepsilon'' = \operatorname{Im} \varepsilon = -\frac{4\pi^2 e^2}{mk^2} \int \mathbf{k} \frac{\partial f}{\partial \mathbf{v}} \delta(\omega - \mathbf{k}\mathbf{v})\, d\mathbf{v} \quad \text{(II. 63)}$$

we obtain

$$S_{ef} = \frac{2e^4}{m^2} \int \frac{\delta(\mathbf{k}\mathbf{v} - \mathbf{k}'\mathbf{v}')}{k^4 |\varepsilon(\mathbf{k}, \mathbf{k}\mathbf{v})|^2} \left\{ f(\mathbf{v}) \mathbf{k} \frac{\partial f(\mathbf{v}')}{\partial \mathbf{v}'} - f(\mathbf{v}') \mathbf{k} \frac{\partial f}{\partial \mathbf{v}} \right\} d\mathbf{k}\, d\mathbf{v}' \quad \text{(II. 64)}$$

In the region $kD > 1$ (D being the Debye screening radius), the dielectric permeability can be set equal to unity, and the integral (II.64) may be shown to reduce to the collision term in the Landau form (where the integral has to be cut off at the upper limit for $k \sim 1/\rho_0$, where ρ_0 is the minimum separation of the particles in the case of binary collisions).

In the form (II.64) the collision term has been obtained in refs. (50), (51), (55) (see also (53, 66–69, 214, 215)). It represents both binary collisions with impact parameter less than D, and also the interaction through longitudinal waves. It has been shown by Davidov (52) that in a plasma approximating to equilibrium, the contribution to the collision term due to the Langmuir waves is only about one order of magnitude smaller than the contribution due to the binary collisions. In a non-equilibrium plasma the contribution due to the oscillations may be considerably larger. For instance, Gorbunov and Silin (54) have shown that in a strongly non-isothermal plasma $T_e/T_i < m_e/m_i$ the interaction due to ion acoustic waves predominates over the binary collisions. Another case where the interaction through the waves is larger than the collisional interaction has been considered in refs.

(44), (45); this is the interaction between the electrons and the cyclotron radiation in a strong magnetic field.

As we approach instability the thermal noise level increases, and consequently in (II.61) the second term becomes more important, which in the quasi-linear approximation leads to the diffusion of the particles in velocity space. In this case, and also in the case of a weak instability of the plasma, it is more convenient to consider an initial-value problem, so that in the kinetic wave eqn. (II.57), we have to take into account the term containing the time derivative of the oscillation intensity, and in the expression for the collision term S_{ef} it is necessary to consider the adiabatic interaction of the particles with the waves, i.e. the term with the derivative $\frac{\partial I_{\mathbf{k}}}{\partial t}$. In other words, we arrive at the quasi-linear approximation considered in Section 3 of Chapter I.

For large oscillation amplitudes, when the plasma goes over into a turbulent motion, it is necessary to consider the non-linear terms in the kinetic wave eqn. (II.57) and in the wave interaction term (II.59). Some examples of such processes will also be considered in Chapter IV. We shall see that in many cases low frequency oscillations can be excited in the plasma by a longitudinal or transverse electron current. When the phase velocity of these oscillations parallel to the magnetic field is smaller than the thermal velocity of the electrons, a resonant interaction may occur between the electrons and the waves, and for the electrons it is then sufficient to consider only the quasi-linear terms. For the ions, the thermal velocity of which may be considerably smaller than the longitudinal phase velocity of the waves, it is necessary to consider the non-linear terms. We then find that the energy and momentum of the electrons is transferred by a resonance mechanism to the waves, and is then absorbed by the ions due to the non-linear damping of the combination waves. Processes of this type may give rise to an anomalous resistivity of the plasma, which determines both the anomalous diffusion of the plasma across the magnetic field and its turbulent heating.

III. METHODS OF CONSIDERING STRONG TURBULENCE

1. The Weak Coupling Approximation

(a) *Wave Equations in Weak Coupling Conditions*

Hitherto we have assumed any interaction between waves to be small and, strictly speaking, infinitely small. Let us now try to examine what occurs as the matrix element of interaction increases. We shall again consider the model eqn. (II.1) which we write in the form

$$(\omega-\omega_{\mathbf{k}})C_{\mathbf{k}\omega}=\int V_{\mathbf{k}\omega,\,\mathbf{k}'\omega'}C_{\mathbf{k}'\omega'}C_{\mathbf{k}-\mathbf{k}',\,\omega-\omega'}\,d\mathbf{k}'\,d\omega' \qquad \text{(III. 1)}$$

where the frequency $\omega_{\mathbf{k}}$ is now complex.

Suppose the matrix element of interaction $V_{\mathbf{k}\omega,\,\mathbf{k}'\omega'}$, though remaining small, increases, approximating in order of magnitude to unity. The interaction between the separate waves broadens out the frequency spectrum of the oscillations and in the limit of strong turbulence the dependence of $I_{\mathbf{k}\omega}$ on the frequency no longer bears any resemblance to a δ-function (see Fig. 12).

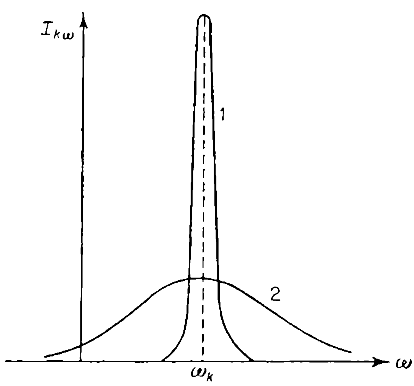

Fig. 12. Spectral functions for weak (1) and strong (2) turbulence

We cannot, therefore, use the kinetic equation in the form (II.12), but as long as the matrix element remains less than unity, we can use the weak coupling method as an approximation even for turbulence which is not weak.

It should be noted that according to eqn. (III.1), a separate mode $\mathbf{k}$, ω interacts only with two other modes $\mathbf{k}'$ ω' and $\mathbf{k}''$ ω''. Since the number of modes is large and the amplitude of the oscillations of an individual mode is determined by its interaction with all other modes, it might be thought that the interaction of the wave $\mathbf{k}$, ω, with each separate wave $\mathbf{k}'$, ω', would be comparatively small even for $V \sim 1$. Moreover, as we have seen further

above, one of the principal non-linear effects is an additional damping, which is implicitly determined by the right hand side of (III.1). It is natural to isolate this part of the interaction earlier from the right hand side by writing equation (III.1) in the form

$$(\omega-\omega_{\mathbf{k}}+\eta_{\mathbf{k}\omega})C_{\mathbf{k}\omega} = \eta_{\mathbf{k}\omega}C_{\mathbf{k}\omega} + \int V_{\mathbf{k}\omega,\mathbf{k}'\omega'}C_{\mathbf{k}'\omega'}C_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\,d\mathbf{k}'\,d\omega' \qquad \text{(III. 2)}$$

The terms $\eta_{\mathbf{k}\omega}\ C_{\mathbf{k}\omega}$ which we have added to both sides of this equation represent the part of the non-linear interaction proportional to $C_{\mathbf{k}\omega}$. In the right hand side of eqn. (III.2), from which we have thus removed the self-action (i.e. the damping) of each mode, only the input from the beat interaction of different modes remains. These inputs will be considered small, which is justified for $V < 1$. Accordingly we again put $C_{\mathbf{k}\omega} = C^{(0)}_{\mathbf{k}\omega}+C^{(1)}_{\mathbf{k}\omega}$ where $C^{(1)}_{\mathbf{k}\omega} \ll C^{(0)}_{\mathbf{k}\omega}$. To determine the amplitude of the induced oscillations $C^{(1)}_{\mathbf{k}\omega}$ we need only the non-linear term on the right hand side of eqn. (III.2) and we obtain

$$C^{(1)}_{\mathbf{k}\omega} = (\omega-\omega_{\mathbf{k}}+\eta_{\mathbf{k}\omega})^{-1}\int V_{\mathbf{k},\omega\,\mathbf{k}'\omega'}C^{(0)}_{\mathbf{k}'\omega'}C^{(0)}_{\mathbf{k}''\omega''}d\mathbf{k}'d\omega' \qquad \text{(III. 3)}$$

We now multiply eqn. (III.2) by $C^{*}_{\mathbf{k}\omega}$ and average the result over the statistical ensemble, assuming that the amplitudes $C^{(0)}_{\mathbf{k}\omega}$ with different $\mathbf{k}\omega$ are statistically independent. We now substitute $C_{\mathbf{k}\omega} = C^{(0)}_{\mathbf{k}\omega}+C^{(1)}_{\mathbf{k}\omega}$ into the non-linear term, and just as in the case of (II.6), this term reduces to a sum of three, two of which are proportional to $I_{\mathbf{k}\omega}$, and a third to the integral of the product $I_{\mathbf{k}'\omega'}\ I_{\mathbf{k}''\omega''}$. Defining the quantity $\eta_{\mathbf{k}\omega}$ so as to eliminate the terms proportional to $I_{\mathbf{k}\omega}$, we obtain the following system of two integral eqns. (III.4) and (III.5)

$$|\omega-\omega_{\mathbf{k}}+\eta_{\mathbf{k}\omega}|^2 I_{\mathbf{k}\omega} = \tfrac{1}{2}\int |v_{\mathbf{k}\omega,\mathbf{k}'\omega'}|^2 I_{\mathbf{k}'\omega'}I_{\mathbf{k}''\omega''}\,d\mathbf{k}'\,d\omega' \qquad \text{(III. 4)}$$

$$\eta_{\mathbf{k}\omega} = \int \frac{v_{\mathbf{k}\omega,\mathbf{k}''\omega''}v_{\mathbf{k}''\omega'',\mathbf{k}\omega}}{\omega''-\omega_{\mathbf{k}''}+\eta_{\mathbf{k}''\omega''}}I_{\mathbf{k}'\omega'}\,d\mathbf{k}'\,d\omega' \qquad \text{(III. 5)}$$

where

$$v_{\mathbf{k}\omega,\mathbf{k}'\omega'} = V_{\mathbf{k}\omega,\mathbf{k}'\omega'} + V_{\mathbf{k}\omega,\mathbf{k}''\omega''}$$

We now introduce $S_{\mathbf{k}\omega} = (\omega-\omega_{\mathbf{k}}+\eta_{\mathbf{k}\omega})^{-1}$, and eliminating $\eta_{\mathbf{k}\omega}$, obtain

$$I_{\mathbf{k}\omega} = \tfrac{1}{2}|S_{\mathbf{k}\omega}|^2\int |v_{\mathbf{k}\omega,\mathbf{k}'\omega'}|^2 I_{\mathbf{k}'\omega'}I_{\mathbf{k}''\omega''}\,d\mathbf{k}'\,d\omega' \qquad \text{(III. 6)}$$

$$S_{\mathbf{k}\omega} = S^0_{\mathbf{k}\omega} - S^0_{\mathbf{k}\omega}S_{\mathbf{k}\omega}\int S_{\mathbf{k}''\omega''}v_{\mathbf{k}\omega,\mathbf{k}''\omega''}v_{\mathbf{k}''\omega'',\mathbf{k}\omega}I_{\mathbf{k}'\omega'}\,d\mathbf{k}'\,d\omega' \qquad \text{(III. 7)}$$

where $S^0_{\mathbf{k}\omega} = (\omega-\omega_{\mathbf{k}})^{-1}$.

The quantities $S_{\mathbf{k}\omega}$ and $S^0_{\mathbf{k}\omega}$ have a simple physical meaning which can be seen by adding a small external source $f_{\mathbf{k}\omega}$ to the right hand side of (III.2). By repeating the argument, it may be verified that $S_{\mathbf{k}\omega}$ represents a Green's function which describes the response of the turbulent medium to a small " force " $f_{\mathbf{k}\omega}$, while $S^0_{\mathbf{k}\omega}$ represents the same Green's function in the linear approximation.

In the theory of fluid turbulence, equations of the form of (III.6) and (III.7) were obtained by Kraichnan (18), and Wild (56) has shown that these equations can be obtained from partial summation of terms in perturbation theory. To obtain Kraichnan's equation we write the Navier-Stokes equation in Fourier representation

$$(\omega + i\nu k^2)u_{\mathbf{k}\omega} - \mathbf{k}p_{\mathbf{k}\omega} - \int (\mathbf{u}_{\mathbf{k}'\omega'}\mathbf{k}'')\mathbf{u}_{\mathbf{k}''\omega''}\,d\mathbf{k}'\,d\omega' = f_{\mathbf{k}\omega} \tag{III. 8}$$

where $p_{\mathbf{k}\omega}$ is the pressure, $f_{\mathbf{k}\omega}$ the external force, ν the coefficient of kinematic viscosity. Using the incompressibility condition $\mathbf{k}\mathbf{u}_{\mathbf{k}\omega} = 0$, and assuming without loss of generality $\mathbf{k}f_{\mathbf{k}\omega} = 0$, we can eliminate the pressure from eqn. (III.8) and obtain

$$(\omega + i\nu k^2)\mathbf{u}_{\mathbf{k}\omega} = f_{\mathbf{k}\omega} + \int (\mathbf{u}_{\mathbf{k}'\omega'}\mathbf{k})\left\{\mathbf{u}_{\mathbf{k}''\omega''} - \frac{\mathbf{k}}{k^2}(\mathbf{k}\mathbf{u}_{\mathbf{k}''\omega''})\right\} d\mathbf{k}'\,d\omega' \tag{III. 9}$$

Repeating the preceding arguments and averaging the result over angles, assuming $f_{\mathbf{k}\omega}$ to be isotropic, we arrive at the following system of integral equations

$$I_{\mathbf{k}\omega} = |S_{\mathbf{k}\omega}|^2 q_{\mathbf{k}\omega} + \tfrac{1}{2}|S_{\mathbf{k}\omega}|^2 \int k^2 a(\mathbf{k}, \mathbf{k}'')I_{\mathbf{k}'\omega'}I_{\mathbf{k}''\omega''}\,d\mathbf{k}'\,d\omega' \tag{III. 10}$$

$$\eta_{\mathbf{k}\omega} = -\int \frac{k^2 b(\mathbf{k}, \mathbf{k}'')I_{\mathbf{k}'\omega'}}{\omega'' + i\nu k''^2 + \eta_{\mathbf{k}''\omega''}}\,d\mathbf{k}'\,d\omega' \tag{III. 11}$$

where $I_{\mathbf{k}\omega}$ is the spectral function defined by

$$I_{\mathbf{k}\omega}\delta(\omega - \omega')\delta(\mathbf{k} - \mathbf{k}') = \langle \mathbf{u}_{\mathbf{k}\omega}\mathbf{u}_{\mathbf{k}'\omega'}\rangle$$

$$S_{\mathbf{k}\omega} = (\omega + i\nu k^2 + \eta_{\mathbf{k}\omega})^{-1}$$

$$q_{\mathbf{k}\omega}\delta(\omega - \omega')\ddot{o}(\mathbf{k} - \mathbf{k}') = \langle f_{\mathbf{k}\omega}f_{\mathbf{k}'\omega'}\rangle$$

and the matrix elements $a(\mathbf{k}, \mathbf{k}'')$ and $b(\mathbf{k}, \mathbf{k}'')$ are defined (see refs. (18, 56)) by the relations

$$a(\mathbf{k}, \mathbf{k}'') = \frac{1}{2}\left[1 - 2\frac{(\mathbf{k}\mathbf{k}')^2(\mathbf{k}\mathbf{k}'')^2}{k^4 k'^2 k''^2} + \frac{(\mathbf{k}\mathbf{k}')(\mathbf{k}\mathbf{k}'')(\mathbf{k}'\mathbf{k}'')}{k^2 k'^2 k''^2}\right] \tag{III. 12}$$

$$b(\mathbf{k}, \mathbf{k}'') = \frac{(\mathbf{k}\ \ ')^3}{k^4 k'^2} - \frac{(\mathbf{k}'\mathbf{k}'')(\mathbf{k}\mathbf{k}'')}{k^2 k''^2} \tag{III. 13}$$

These equations represent a Fourier transform of Kraichnan's equations. It has been shown by Kraichnan that they lead to an incorrect asymptotic behaviour of the spectral function for $k \to \infty$; instead of the well-known Kolomogorov law $I_k dk \sim k^{-5/3}\,dk$ we obtain the spectrum $I_k dk \sim k^{-3/2}\,dk$. We shall show below that this occurs because in the weak coupling approximation used here, the adiabatic interaction of the distant harmonics is not separated out.

(b) *Weak Coupling in Kinetics*

We shall show by the example of electron Langmuir oscillations how the weak coupling equations can be set up for strong turbulence in a collisionless

plasma. We take as the initial equations

$$\varphi_{\mathbf{k}\omega} = -\frac{4\pi e}{k^2}\int f_{\mathbf{k}\omega}\,d\mathbf{v} \tag{III. 14}$$

$$(\omega - \mathbf{kv})f_{\mathbf{k}\omega} = \frac{e}{m}\varphi_{\mathbf{k}\omega}\mathbf{k}\frac{\partial f}{\partial \mathbf{v}} + \frac{e}{m}\frac{\partial}{\partial \mathbf{v}}\int \mathbf{k}'\{\varphi_{\mathbf{k}'\omega'}f_{\mathbf{k}''\omega''} - \langle\varphi_{\mathbf{k}'\omega'}f_{\mathbf{k}''\omega''}\rangle\}\,d\mathbf{k}'\,d\omega' \tag{III. 15}$$

We again represent $f_{\mathbf{k}\omega}$ and $\varphi_{\mathbf{k}\omega}$ in the form $f^{(0)}_{\mathbf{k}\omega} + f^{(1)}_{\mathbf{k}\omega}$, $\varphi^{(0)}_{\mathbf{k}\omega} + \varphi^{(1)}_{\mathbf{k}\omega}$, where the superscript (1) denotes the part representing the induced oscillations. When these forms for f and φ are substituted in the non-linear terms, we shall just as before be able to extract " self-action " terms proportional respectively to $f_{\mathbf{k}\omega}$ and $\varphi_{\mathbf{k}\omega}$, and it is again natural to separate out these terms. Because of the non-linearity these terms are not proportional to one another, or in other words their ratio is a random value. We therefore write the self-action terms as a linear combination of $f_{\mathbf{k}\omega}$ and $\varphi_{\mathbf{k}\omega}$: $\eta_{\mathbf{k}\omega}f_{\mathbf{k}\omega} + \xi_{\mathbf{k}\omega}\,\varphi_{\mathbf{k}\omega}$, where $\eta_{\mathbf{k}\omega}$ is an operator acting on the variable v, and $\xi_{\mathbf{k}\omega}$ is a function of velocity. As a result we obtain

$$\varphi^{(1)}_{\mathbf{k}\omega} = -\frac{4\pi e}{k^2}\int f^{(1)}_{\mathbf{k}\omega}\,d\mathbf{v} \tag{III. 16}$$

$$(\omega - \mathbf{kv} + \eta_{\mathbf{k}\omega})f^{(1)}_{\mathbf{k}\omega} - \xi_{\mathbf{k}\omega}\varphi^{(1)}_{\mathbf{k}\omega} - \frac{e}{m}\varphi^{(1)}_{\mathbf{k}\omega}\mathbf{k}\frac{\partial f}{\partial \mathbf{v}} = \frac{e}{m}\frac{\partial}{\partial \mathbf{v}}\int \mathbf{k}'\varphi^{(0)}_{\mathbf{k}'\omega'}f^{(0)}_{\mathbf{k}''\omega''}\,d\mathbf{k}'\,d\omega' \tag{III. 17}$$

From these equations we express $f^{(1)}_{\mathbf{k}\omega}$ and $\varphi^{(1)}_{\mathbf{k}\omega}$ in terms of an integral of $\varphi^{(0)}_{\mathbf{k}'\omega'}\,f^{(0)}_{\mathbf{k}''\omega''}$ and substitute the resulting expressions in the non-linear term of eqn. (III.15). Then, multiplying eqns. (III.14), (III.15) by $\varphi^{*}_{\mathbf{k}\omega}$ and averaging over the random phases of $f^{(0)}_{\mathbf{k}\omega}$, $\varphi^{(0)}_{\mathbf{k}\omega}$, we obtain, dropping the superscript (0)

$$I_{\mathbf{k}\omega} = -\frac{4\pi e}{k^2}\int P_{\mathbf{k}\omega}\,d\mathbf{v} \tag{III. 18}$$

$$\begin{aligned}(\omega - \mathbf{kv})P_{\mathbf{k}\omega} = {} & \frac{e}{m}\mathbf{k}\frac{\partial f}{\partial \mathbf{v}}I_{\mathbf{k}\omega} - \frac{e}{m}\frac{\partial}{\partial \mathbf{v}} \times \\ & \times \int \mathbf{k}'\tilde{\mathbf{g}}_{\mathbf{k}''\omega''}(\mathbf{k}P_{-\mathbf{k}'-\omega'}I_{\mathbf{k}\omega} - \mathbf{k}'P_{\mathbf{k}\omega}I_{\mathbf{k}'\omega'})\,d\mathbf{k}'\,d\omega' + I_{\mathbf{k}\omega}\frac{e}{m}\frac{\partial}{\partial \mathbf{v}} \times \\ & \times \int \mathbf{k}'(\tilde{\mathbf{g}}_{\mathbf{k}''\omega''}\mathbf{k}''f)\left\{\frac{\tilde{\nu}_{\mathbf{k}''\omega'',\,\mathbf{k}\omega}}{\tilde{\varepsilon}(\mathbf{k}''\omega'')}I_{\mathbf{k}'\omega'} + \frac{\tilde{\nu}_{\mathbf{k}'\omega',\,\mathbf{k}\omega}}{\tilde{\varepsilon}(\mathbf{k}'\omega')}I_{\mathbf{k}''\omega''}\right\}d\mathbf{k}'\,d\omega' + \\ & + \frac{e}{m}\frac{\partial}{\partial \mathbf{v}}\int \mathbf{k}'(\tilde{\mathbf{g}}_{\mathbf{k}''\omega''}\mathbf{k}''f)\frac{\tilde{\nu}^{*}_{\mathbf{k}\omega,\,\mathbf{k}'\omega'}}{\tilde{\varepsilon}^{*}(\mathbf{k}\omega)}I_{\mathbf{k}'\omega'}I_{\mathbf{k}''\omega''}\,d\mathbf{k}'\,d\omega'\end{aligned} \tag{III. 19}$$

where

$$\tilde{\mathfrak{g}}_{\mathbf{k}\omega} = (\omega - \mathbf{kv} + \eta_{\mathbf{k}\omega})^{-1} \frac{e}{m} \frac{\partial}{\partial \mathbf{v}}$$

$$\tilde{\varepsilon}(\mathbf{k}, \omega) = 1 + \frac{4\pi e^2}{mk^2} \int \frac{\mathbf{k}\dfrac{\partial f}{\partial \mathbf{v}}}{\omega - \mathbf{kv} + \eta_{\mathbf{k}\omega}} d\mathbf{v} + \frac{4\pi e}{k^2} \int \frac{\xi_{\mathbf{k}\omega}\, d\mathbf{v}}{\omega - \mathbf{kv} + \eta_{\mathbf{k}\omega}}$$

$$\tilde{v}_{\mathbf{k}\omega, \mathbf{k}'\omega'} = -\frac{4\pi e}{k^2} \int \{(\tilde{\mathfrak{g}}_{\mathbf{k}\omega}\mathbf{k}')(\tilde{\mathfrak{g}}_{\mathbf{k}''\omega''}\mathbf{k}''f) + (\tilde{\mathfrak{g}}_{\mathbf{k}\omega}\mathbf{k}'')(\tilde{\mathfrak{g}}_{\mathbf{k}'\omega'}\mathbf{k}'f)\}\, d\mathbf{v} \quad \text{(III. 20)}$$

It can be seen from (III.19) that in this approximation the non-linear term reduces to a sum of two components, of which the first is proportional to $P_{\mathbf{k}\omega}$, and the second originates from averaging in the following form $\langle \varphi^* \varphi \rangle$. Identifying these correspondingly with $\eta_{\mathbf{k}\omega} P_{\mathbf{k}\omega}$ and $\xi_{\mathbf{k}\omega} I_{\mathbf{k}\omega}$, gives

$$\eta_{\mathbf{k}\omega} \mu_{\mathbf{k}\omega} = \frac{e}{m} \frac{\partial}{\partial v} \int \mathbf{k}'(\mathbf{k}'\tilde{\mathfrak{g}}_{\mathbf{k}''\omega''})\mu_{\mathbf{k}\omega} I_{\mathbf{k}'\omega'}\, d\mathbf{k}'\, d\omega' \quad \text{(III. 21)}$$

$$\xi_{\mathbf{k}\omega} = \frac{e}{m} \frac{\partial}{\partial v} \int (\mathbf{k}'\tilde{\mathfrak{g}}_{\mathbf{k}''\omega''})\mathbf{k} P_{-\mathbf{k}'-\omega'}\, d\mathbf{k}'\, d\omega' +$$

$$+ \frac{e}{m} \frac{\partial}{\partial v} \int k'(\tilde{\mathfrak{g}}_{\mathbf{k}''\omega''}\mathbf{k}''f) \left\{ \frac{\tilde{v}_{\mathbf{k}''\omega'', \mathbf{k}\omega}}{\tilde{\varepsilon}(\mathbf{k}''\omega'')} I_{\mathbf{k}'\omega'} + \frac{\tilde{v}_{\mathbf{k}'\omega', \mathbf{k}\omega}}{\varepsilon(\mathbf{k}'\omega')} I_{\mathbf{k}''\omega''} \right\} d\mathbf{k}'\, d\omega' \quad \text{(III. 21a)}$$

where $\mu_{\mathbf{k}\omega} = P_{\mathbf{k}\omega}/I_{\mathbf{k}\omega}$.

Considering these relations and substituting the expression for $P_{\mathbf{k}\omega}$ from (III.19) in eqn. (III.18), we obtain the first integral equation in the form

$$|\tilde{\varepsilon}(\mathbf{k}, \omega)|^2 I_{\mathbf{k}\omega} = \tfrac{1}{2} \int |\tilde{v}_{\mathbf{k}\omega, \mathbf{k}'\omega'}|^2 I_{\mathbf{k}'\omega'} I_{\mathbf{k}''\omega''}\, d\mathbf{k}'\, d\omega' \quad \text{(III. 22)}$$

Equation (III.21) can be used as the second equation and the third is obtained by substituting the expression for $\xi_{\mathbf{k}\omega}$ (III.21a) in eqn. (III.20)

$$\tilde{\varepsilon}(\mathbf{k}, \omega) = 1 + \frac{4\pi e}{k^2} \int (\tilde{\mathfrak{g}}_{\mathbf{k}\omega}\mathbf{k}) f d\mathbf{v} + \frac{4\pi e}{k^2} \int (\mathbf{k}\tilde{\mathfrak{g}}_{\mathbf{k}\omega})(\mathbf{k}'\tilde{\mathfrak{g}}_{\mathbf{k}''\omega''}) P_{-\mathbf{k}'-\omega'}\, d\mathbf{k}'\, d\omega'\, d\mathbf{v} +$$

$$+ \frac{4\pi e}{k^2} \int \frac{\tilde{v}_{\mathbf{k}\omega, \mathbf{k}''\omega''}\, \tilde{v}_{\mathbf{k}''\omega'', \mathbf{k}\omega}}{\tilde{\varepsilon}(\mathbf{k}''\omega'')} I_{\mathbf{k}'\omega'}\, d\mathbf{k}'\, d\omega' \quad \text{(III. 23)}$$

If we write approximately $\mu_{\mathbf{k}\omega} = \tilde{\mathfrak{g}}_{\mathbf{k}\omega}\mathbf{k}f$, eqns. (III.21), (III.22), (III.23) constitute a complete system of equations for the three unknown quantities $\eta_{\mathbf{k}\omega}$, $\tilde{\varepsilon}(\mathbf{k}\omega)$, and $I_{\mathbf{k}\omega}$. Since $\eta_{\mathbf{k}\omega}$ is an operator, these equations are symbolical, and to explain their meaning it is necessary to perform an expansion into a power series with respect to $I_{\mathbf{k}\omega}$. A series expansion in the amplitude of oscillations and the selective summing of this series is also necessary for the rigorous justification of the above equations, which we have obtained essentially from semi-intuitive considerations. (It has been shown by Mikhailovskiĭ that in the case of a plasma in a strong magnetic field integral equations of the form (III.21), (III.22) can in fact be obtained by selective summing, similar to Wild's summation, which as a first approximation leads to Kraichnan's equations.)

(c) *Resonance and Adiabatic Wave Interaction*

In the arguments given above we implicitly allowed only resonant interactions between waves. However, the interaction of modes of very different wavelengths does not necessarily have to be of the resonance type. (Both here and below we are discussing only that part of the interaction which leads to secular terms, i.e. to a damping of the modes, and not the part whose effect is to shift their characteristic frequencies.)

In order to analyse the character of the interaction rather more carefully, we shall again consider the model eqn. (III.1). Suppose for simplicity that the frequency $\omega_{\mathbf{k}}$ increases monotonically with k, and the ratio of the growth rate to the frequency, $\gamma_{\mathbf{k}}/\omega_{\mathbf{k}}$, is small. As we explained earlier, in a stationary turbulent state the damping of each mode is compensated by the input due to the beat interaction. The wave $\mathbf{k}$, ω then exists for a time $\sim \gamma^{-1}$ and occupies a region in space of size $\sim \dfrac{\omega}{\gamma k}$. During a period of the order γ^{-1}, an individual wave disappears completely and is replaced by another wave, which originates from the beat interaction and consequently cannot be correlated with the first. Although the mode may initially be localised, it in general spreads out as time passes, to fill a region of space whose characteristic size L will be of the order of the distance through which the wave propagates during its lifetime, i.e. $L \sim \gamma^{-1} \dfrac{d\omega}{dk} \sim \dfrac{\omega}{\gamma k}$.

Thus the state of turbulent motion of a continuous medium must be regarded as a system of many wave packets. For $\gamma/\omega \ll 1$ these packets exist for a very long time and are very extended, so that one can describe them as waves which are almost completely unlocalised in space. But as γ/ω increases, we must explicitly consider that the elements of the turbulent motion are not the Fourier components, but wave packets. In other words, for a finite γ/ω, nearby Fourier components can no longer be considered weakly correlated.

We can put the matter in another way as follows. Consider the region of wave numbers of order k. Since the waves in this region exist for a period $t \sim \gamma_{\mathbf{k}}^{-1}$, when we average the quadratic terms over time it is sufficient to cover times of the order of $\gamma_{\mathbf{k}}^{-1}$. All slower oscillations, with frequencies $\omega' \lesssim \gamma_{\mathbf{k}}$, can be considered fixed during this period of time and need not be averaged. These modes can be regarded as forming an inhomogeneity of the background. For short waves, this inhomogeneity may be considered in the quasi-classical approximation discussed earlier. As we have seen, the wave packets are deformed and the wavelength changed as they propagate through such an inhomogeneity. As a result of these changes, the wave packets move about in wave number space, which leads to a strong correlation of nearby Fourier components, which now describe essentially one and the same wave packet.

Thus the interaction between the wave $\mathbf{k}$, ω, and the slow wave $\mathbf{k}'\omega'$ leads to an adiabatic variation of the wave vector $\mathbf{k}$ and of the frequency ω of the

wave packet under consideration. This interaction cannot be considered as the resonant input to the wave $\mathbf{k}$, ω from the nearby wave $\mathbf{k}'' = \mathbf{k}-\mathbf{k}'$, $\omega'' = \omega-\omega'$, because the Fourier components $C_{\mathbf{k}\omega}$ and $C_{\mathbf{k}''\omega''}$ describe the same wave packet and cannot be considered independently of one another.

Thus, for finite γ/ω we must make a distinction between resonant and adiabatic wave interactions. To bring in this distinction we must change from the Fourier representation to a representation in terms of the wave packets. For each packet with a mean frequency ω and mean wave vector k the region of integration with respect to $\mathbf{k}'$, ω' in the non-linear term is conveniently broken up into three parts: (1) the principal region where $k' \sim k$, $\omega' \sim \omega$, (2) the long wave region where $\omega' \sim \gamma$ and (3) the short wave region where $\gamma' \sim \omega$ (γ/ω is assumed small).

For small γ/ω the spreading of the wave packets in the principal region can be neglected. In this case we can use eqn. (III.1) for the Fourier components. In the regions (2) and (3) the quasi-classical approximation can be used, i.e. an expansion in k'/k, ω'/ω and k/k', ω/ω' respectively. Limiting the treatment to the first (or more precisely to the zero) approximation, we obtain

$$(\omega-\omega_{\mathbf{k}})C_{\mathbf{k}\omega} \cong \int\limits_{(1)} V_{\mathbf{k}\omega,\mathbf{k}'\omega'}C_{\mathbf{k}'\omega'}C_{\mathbf{k}''\omega''}\,d\mathbf{k}'\,d\omega' + \\ + C_{\mathbf{k}\omega}\int\limits_{(2)} (V_{\mathbf{k}\omega,\mathbf{k}'\omega'} + V_{\mathbf{k}\omega,\mathbf{k}''\omega''})C_{\mathbf{k}'\omega'}\,d\mathbf{k}'\,d\omega' + \\ + \int\limits_{(3)} V_{\mathbf{k}\omega,\mathbf{k}'\omega'}|C_{\mathbf{k}'\omega'}|^2\,d\mathbf{k}'\,d\omega' \quad \text{(III. 24)}$$

where for simplicity we have omitted the exponent $\exp(-i\omega't+i\mathbf{k}'\mathbf{r})$ in the integral over the region (2), assuming $C_{\mathbf{k}'\omega'}$ to be a slowly varying function of $\mathbf{r}$, t.

In (III.24) the last integral term can be omitted, because when we divided the equations for C into two—for the averaged function and the oscillating part—we should have included it in the equation for the averaged function. The second integral on the right hand side of (III.24) is proportional to $C_{\mathbf{k}\omega}$ and consequently it leads only to a shift of the characteristic frequency. In the region (1) this shift can be considered approximately constant. If the matrix element $V_{\mathbf{k}\omega,\mathbf{k}'\omega'}$ depends only on the frequency difference $\omega-\omega'$, or is independent of ω, ω', which is often the case, the frequency shift can be eliminated by changing the variable appropriately. The final result is that we recover eqn. (III.1), with the difference that the range of integration in the non-linear term is restricted to cover the region (1). It follows that separate consideration of the adiabatic interaction is important only for wave numbers sufficiently large that the lower limit of integration of region (1) is greater than the main turbulence scale k_0, ω_0.

It is now obvious why the simple weak coupling approximation led Kraichnan to an incorrect spectrum: this approximation over-estimates the part played by the large-scale fluctuations, which is in fact no more than the convection of higher modes which are deformed adiabatically in the process.

If we make the corresponding change in the weak coupling eqns. (III.10), (III.11) by setting the lower limit of the integrations over $\mathbf{k}'$, ω' equal to $\xi\mathbf{k}$, $\xi\omega$, say, respectively, where ξ is a small constant number (say $\xi \sim \frac{1}{3}$), these equations become self-modelling and lead to the correct spectral form $I_k dk \sim k^{-\frac{5}{3}}\, dk$. However, it is obviously sufficient to use the simple weak coupling equations for modes in the region of the main scale.

(d) *Improved Weak Coupling Approximation*

To improve our understanding of the interaction between modes in a turbulent medium, we shall here give explicitly a series expansion of the second term on the right hand side of eqn. (III.24), to take into account the adiabatic distortion of the modes under consideration by the long wave perturbations. Assuming for simplicity the matrix element to be independent of the frequency, we obtain

$$(\omega-\omega_{\mathbf{k}}-\omega_l)C_{\mathbf{k}\omega} \cong \int_{(1)} V_{\mathbf{k}\mathbf{k}'}C_{\mathbf{k}'\omega'}C_{\mathbf{k}-\mathbf{k}',\,\omega-\omega'}\,d\mathbf{k}'\,d\omega' -$$

$$-\frac{\partial C_{\mathbf{k}\omega}}{\partial \mathbf{k}}\int_{(2)} \mathbf{k}'v_{\mathbf{k}\mathbf{k}'}C_{\mathbf{k}'\omega'}\,d\mathbf{k}'\,d\omega' - \frac{\partial C_{\mathbf{k}\omega}}{\partial \omega}\int_{(2)} \omega' v_{\mathbf{k}\mathbf{k}'}C_{\mathbf{k}'\omega'}\,d\mathbf{k}'\,d\omega' +$$

$$+\tfrac{1}{2}\int_{(3)} v_{\mathbf{k}\mathbf{k}'}\langle C_{\mathbf{k}'\omega'}C_{\mathbf{k}''\omega''}\rangle_s\,d\mathbf{k}'\,d\omega'\,d\mathbf{k}''\,d\omega'' \qquad \text{(III. 25)}$$

where

$$v_{\mathbf{k}\mathbf{k}'} = V_{\mathbf{k}\mathbf{k}'}+V_{\mathbf{k},\,\mathbf{k}-\mathbf{k}'}, \qquad \omega_l = \int_{(2)} v_{\mathbf{k}\mathbf{k}'}C_{\mathbf{k}'\omega'}\,d\mathbf{k}'\,d\omega' \qquad \text{(III. 26)}$$

The last term in (III.25) has been written as an average over high frequency oscillations, denoted by $\langle\ldots\rangle_s$, which is permissible since only the secular effect of the oscillations in the high frequency region (3) can be important for the modes under consideration.

The second and third terms in the right hand side of (III.25) are proportional to $\dfrac{\partial\omega_l}{\partial\mathbf{r}}$ and $\dfrac{\partial\omega_l}{\partial t}$. They describe the distortion of the wave packet $\mathbf{k}$, ω, due to weak spatial inhomogeneity and its slow variation in time. As we have said, this effect is the same as the distortion of the wave packets in an inhomogeneous medium which we considered earlier (see Section I.3b).

In eqn. (III.25) it is convenient to change the variable to the relative frequencies $\nu = \omega-\omega_l$, with the corresponding amplitudes denoted by $C_{\mathbf{k}\nu}$. It is easily seen with this transformation that the third integral on the right hand side of (III.25) vanishes, since $i\dfrac{\partial C_{\mathbf{k}\nu}}{\partial t} = \left(\nu - \dfrac{\partial\omega_l}{\partial t}\dfrac{\partial}{\partial\nu}\right)C_{\mathbf{k}\nu}$. The remaining terms retain their previous form, apart from the change of the frequencies ω, ω' to ν, ν'.

Using the argument of the previous section, we again separate out from the right hand side of (III.25) the quantity $\eta_{\mathbf{k}\nu}\,C_{\mathbf{k}\nu}$, proportional to $C_{\mathbf{k}\nu}$, and write the amplitude in the form $C_{\mathbf{k}\nu}^{(0)}+C_{\mathbf{k}\nu}^{(1)}$, where $C_{\mathbf{k}\nu}^{(1)}$ describes the induced

oscillations. According to (III.25) the amplitude $C^{(1)}_{\mathbf{k}\nu}$ can be written as a sum of two components, $C^{(1)}_{\mathbf{k}\nu r}+C^{(1)}_{\mathbf{k}\nu l}$ where the first is due to resonance interaction in the region (1) and the second given by

$$C^{(1)}_{\mathbf{k}\nu l} = -(\nu-\omega_{\mathbf{k}}+\eta_{\mathbf{k}\nu})^{-1}\frac{\partial C^{(0)}_{\mathbf{k}\nu}}{\partial \mathbf{k}}\int\limits_{(2)} \mathbf{k}' v_{\mathbf{k}\mathbf{k}'} C^{(0)}_{\mathbf{k}'\omega'}\, d\mathbf{k}'\, d\omega' \qquad \text{(III. 27)}$$

originates from the distortion of the wave packets at the long wavelength inhomogeneities.

Multiplying by $C^{*}_{\mathbf{k}\nu}$ and averaging over the random phases of the amplitudes $C^{(0)}_{\mathbf{k}\nu}$, we obtain, in addition to the contribution due to the resonance interaction, terms of two new types. Two terms of the first type originate from the second on the right hand side of eqn. (III.25). Their sum is proportional to $C^{(1)*}_{\mathbf{k}\nu l}\dfrac{\partial C^{(0)}_{\mathbf{k}\nu}}{\partial \mathbf{k}} + C^{(0)*}_{\mathbf{k}\nu}\dfrac{\partial C^{(1)}_{\mathbf{k}\nu l}}{\partial \mathbf{k}}$. These terms describe the diffusion of the wave packet $\mathbf{k}$, ν in momentum space due to the distortion introduced by the long wave length modes. The value of the corresponding diffusion coefficient will be proportional to the integral over region (2) of the quantity $k'^2 I_{\mathbf{k}'\nu'}$. Since the wave number k' in this region is very small, this diffusion can be neglected.† Additional terms of the second type originate from the last integral in eqn. (III.25), where instead of $C_{\mathbf{k}'\omega'}$ or $C_{\mathbf{k}''\omega''}$ we must put $C^{(1)}_{l}$. Since $C_{\mathbf{k}'\omega'}$ and $C_{\mathbf{k}''\omega''}$ enter perfectly symmetrically, it is sufficient to replace only one of these, say $C_{\mathbf{k}'\omega'}$, and then to double the result. Now for region (3) the long wave region (a region (2′) as it were) is represented by our region (1), and using this the corresponding contribution may be obtained without difficulty. It is proportional to $I_{\mathbf{k}\nu}$, and should therefore be included in the equation for $\eta_{\mathbf{k}\nu}$. We finally obtain

$$\tilde{\eta}_{\mathbf{k}\nu} = -\int\limits_{(1)} \frac{v_{\mathbf{k}\mathbf{k}''} v_{\mathbf{k}''\mathbf{k}} \tilde{I}_{\mathbf{k}'\nu'}}{\nu''-\omega_{\mathbf{k}''}+\tilde{\eta}_{\mathbf{k}''\nu''}}\, d\mathbf{k}'\, d\nu' + \int\limits_{(3)} \frac{v_{\mathbf{k}\mathbf{k}'} v_{\mathbf{k}'\mathbf{k}}}{\nu'-\omega_{\mathbf{k}'}+\tilde{\eta}_{\mathbf{k}'\nu'}}\, \mathbf{k}\frac{\partial \tilde{I}_{\mathbf{k}'\nu'}}{\partial \mathbf{k}'}\, d\mathbf{k}'\, d\nu' \qquad \text{(III. 28)}$$

$$\tilde{I}_{\mathbf{k}\nu} = \tfrac{1}{2}|\tilde{S}_{\mathbf{k}\nu}|^2 \int\limits_{(1)} |v_{\mathbf{k}\mathbf{k}'}|^2 \tilde{I}_{\mathbf{k}'\nu'} \tilde{I}_{\mathbf{k}''\nu''}\, d\mathbf{k}'\, d\nu' \qquad \text{(III. 29)}$$

where $\quad \mathbf{k}'' = \mathbf{k}-\mathbf{k}', \qquad \omega'' = \omega-\omega', \qquad \tilde{S}_{\mathbf{k}\nu} = (\nu-\omega_{\mathbf{k}}+\tilde{\eta}_{\mathbf{k}\nu})^{-1}$

We have marked the intensity $\tilde{I}_{\mathbf{k}\nu}$ and the Green function $\tilde{S}_{\mathbf{k}\nu}$ with a tilde to emphasize again that these values are calculated in a system of co-ordinates moving with the long wave pulsations. The true functions $I_{\mathbf{k}\omega}$, $S_{\mathbf{k}\omega}$ are defined by the relations

$$I_{\mathbf{k}\omega} = \langle \tilde{I}_{\mathbf{k},\,\omega-\omega_l}\rangle_l, \qquad S_{\mathbf{k}\omega} = \langle \tilde{S}_{\mathbf{k},\,\omega-\omega_l}\rangle_l \qquad \text{(III. 30)}$$

which are averaged over the long wave pulsations (we recall that ω_l is still a random value).

† This assertion is no longer valid where for the modes under consideration there is no resonance interaction: in that case the distortion of the wave packets due to the long wavelength modes becomes the dominant effect. Such a situation occurs for instance, in the short wave (viscous) region of ordinary turbulence (see (57) (58)).

According to eqn. (III.26), ω_l can be represented approximately by $\omega_l = v_{\mathbf{kk}} C_l$, where $C_l = \int_{(2)} C_{\mathbf{k}\omega}\, d\mathbf{k}\, d\omega$ is the amplitude of the whole set of long wavelength modes represented by region (2). C_l is a random quantity given by the sum of a large number of individually random and weakly correlated amplitudes; thus the distribution of C_l, and consequently also that of ω_l, is Gaussian. Thus the relations (III.30) can be put in the form

$$I_{\mathbf{k}\omega} = \frac{1}{\pi\omega_0}\int e^{-\frac{\omega_l{}^2}{\omega_0{}^2}} \tilde{I}_{\mathbf{k},\,\omega-\omega_l}\, d\omega_l, \qquad S_{\mathbf{k}\omega} = \frac{1}{\pi\omega_0}\int e^{-\frac{\omega_l{}^2}{\omega_0{}^2}} \tilde{S}_{\mathbf{k},\,\omega-\omega_l}\, d\omega_l \qquad \text{(III. 31)}$$

where

$$\omega_0^2 = 2\langle\omega_l^2\rangle = 2|v_{\mathbf{kk}}|^2 \int_{(2)} I_{\mathbf{k}'\omega'}\, d\mathbf{k}'\, d\omega' \qquad \text{(III. 32)}$$

The calculation of $I_{\mathbf{k}\omega}$, $S_{\mathbf{k}\omega}$ is of course only necessary if we are in fact interested in the temporal correlation of the fluctuations. In a number of problems it is quite sufficient to know only $I_{\mathbf{k}} = \int I_{\mathbf{k}\omega} d\omega = \int \tilde{I}_{\mathbf{k}v} dv$, and we need then only calculate the relative values $\tilde{I}_{\mathbf{k}v}$ and $\tilde{S}_{\mathbf{k}v}$.

Equations (III.28) and (III.29) together with the relations (III.30) constitute the equations of an improved weak coupling approximation. The main contribution to the damping of the waves is supplied by the resonance region, but, according to eqn. (III.27) a small additional damping arises in the short wave region (3). According to eqn. (III.29), the intensity of the oscillations $\tilde{I}_{\mathbf{k}v}$ is determined by the resonance interaction, but there is in addition a small contribution which we omitted, of the type of a diffusion in velocity space, which arises from the interaction with long wave fluctuations in region (2) and also, finally, an altogether insignificant addition may also be supplied by small occasional impulses due to the short wave oscillations of region (3).

If we now reduce the coupling between the oscillations, we see that region (1) expands and in the limit $\gamma/\omega \to 0$ it occupies the whole of $\mathbf{k}$ space apart from isolated regions where the damping is not weak, and the perturbations can reach equilibrium with the remaining fluctuations fairly rapidly. In a plasma, there is a region of this type at short wavelengths $kD > 1$, where all oscillations are damped. This region in particular defines the binary collision term. Because of the weak (logarithmic) dependence of the collisional term (II.64) on the upper cut-off parameter, it is possible to write it inadvertently in the investigation of any given set of oscillations. In fact short wave fluctuations follow the oscillations adiabatically, although from the point of view of the quasi-linear approach (see Section II.2(b)) they must be considered as equivalent to all other oscillations.

2. Phenomenological Approach to the Description of a Strong Turbulence

As we have shown, a quantitative theory only exists for the limit of weak turbulence where a kinetic equation for the waves can be set up. For this theory to be applicable the interaction between the waves must satisfy

several important limitations: the growth rate of small perturbations must be small compared with the frequency, the dispersion relation must be such that the laws of conservation of momentum and energy can be satisfied simultaneously at discrete surfaces in momentum space, the interaction between the waves must provide additional dissipation in each region in momentum space, and so forth. In practice these limitations must fairly often be infringed, and then strong turbulence develops.

Starting with the kinetic wave equation, we have shown above that the transition to strong turbulence leads to integral equations in the weak coupling approximation, in which the resonant and adiabatic interaction must be separated. At present we have no rigorous method of performing this separation and of reducing the integral equations, and the description given by this theory is inevitably only approximate. We have as yet no indication of the accuracy of this approximation.

However, in describing strong turbulence in a plasma we can use an analogy with ordinary turbulence. Here the principal results have been obtained from a purely phenomenological approach. It is natural, therefore, to use this approach in plasma turbulence theory.

In the next section we summarize some of the results of the phenomenological description of ordinary turbulence, and will show later how similar concepts may be used when considering plasma turbulence.

(a) *The Turbulent Jet*

Using the phenomenological approach, the mean velocity profile across a turbulent flow may be obtained without discussing the spectral function. We use the mixing length concept introduced by Prandtl (59), and consider one of the simplest problems, the velocity profile in a submerged turbulent jet. Such a jet is formed when a liquid or gas flows out of an orifice into a medium of the same state. When it emerges from the orifice the velocity of the fluid is nearly constant over the cross-section; as the distance from the orifice increases, the velocity profile becomes more and more deformed until, at a fairly large distance the jet becomes self-modelling, i.e. the velocity profile retains the same form but its width increases in proportion to the distance from the orifice.

For simplicity we consider a flat jet emerging from a slit. The mean velocity depends only on the co-ordinates x parallel to the jet and y transverse to it. The mean pressure in the free jet can be assumed constant, and the longitudinal component of the averaged equation of motion can be written in the form:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y} \qquad \text{(III. 33)}$$

where u and v represent the x- and y-components of the mean velocity respectively, ρ the density of the fluid, $\tau_{xy} = -\rho\langle u'v'\rangle$ is the tangential Reynolds stress arising from the transfer of longitudinal fluctuating momentum $\rho u'$ by the transverse fluctuations v'. The value of v' is of the same order

as that of u', since they are related by the continuity equation. To obtain an estimate of u', Prandtl assumed that the velocity fluctuations arose essentially from the conservation of the mean longitudinal momentum of a fluid element during its transverse displacement. This gives the estimate

$$u' \cong l\left|\frac{\partial u}{\partial y}\right| \qquad \text{(III. 34)}$$

where l is the mean transverse displacement of the fluid elements, which is the so-called " mixing length ". In a free jet the mixing length can be assumed to be approximately constant over the cross-section and proportional to the thickness of the jet, i.e. $l = cx$, where x is the distance from the origin of the jet, and c is a constant. Thus we obtain approximately

$$\tau_{xy} = \pm\rho c^2 x^2 \frac{\partial u}{\partial y}\left|\frac{\partial u}{\partial y}\right| \qquad \text{(III. 35)}$$

and eqn. (III.33) becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \pm 2c^2 x^2 \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} \qquad \text{(III. 36)}$$

This equation and the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{(III. 37)}$$

are a complete set of equations, and determine the mean velocity profile in a turbulent jet. This system contains only one unknown constant c, which can be determined by comparing the theoretically calculated velocity profile with that measured experimentally.

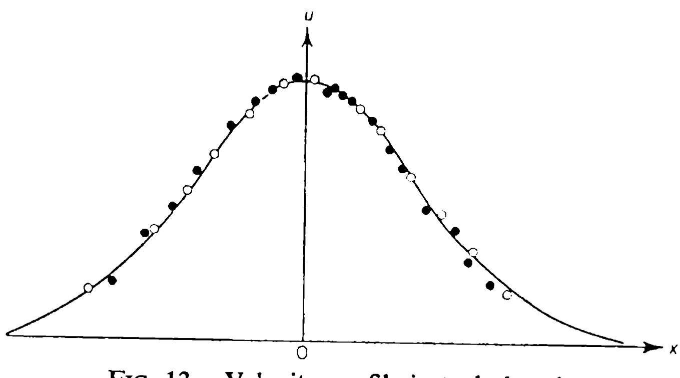

FIG. 13. Velocity profile in turbulent jet

Equations (III.36) and (III.37) were solved by Tollmien (60). Figure 13 shows a comparison between Tollmien's profile and the experimentally measured profile (this figure has been taken from ref. (61)). With an appropriate choice of the constant c the theoretical profile agrees very well with that determined experimentally. The mixing length turns out to be an order smaller than the half thickness of the jet.

While other authors have proposed different hypotheses relating to the character of the momentum transfer across the jet (see, for instance, ref. (61)),

the general validity of Prandtl's description of the process in terms of mixing length theory has remained unchallenged.

(b) *The Kolmogorov-Obukhov Law*

Let us now go on to the problem of the fluctuation spectrum in a turbulent fluid. It is natural to limit the discussion to scales which are considerably smaller than the principal scale, so that it is possible to refer to local properties of the turbulent flow. The theory of such a locally isotropic turbulence was developed by Kolmogorov and Obukhov (63, 64) and later by Heisenberg (65). We shall discuss here only the principal results of this theory, and this in considerably simplified form.

When the Reynolds number is very large, the fluid viscosity is negligible over a wide range of scales of the turbulent motion. Over this range the energy cannot be dissipated, but can only diffuse in wave-number space. This principal region of the spectrum of the turbulent fluctuations is called the inertial range. On the other hand, in the region of lengths, less than some small scale λ_0—the so-called internal scale of the turbulence—the viscosity is dominant. In this region the energy of the motion is dissipated into heat.

According to Kolmogorov, in the inertial range of the spectrum a quasi-equilibrium is established, in which there is a constant energy flux ε through the spectrum into the short wave region $\lambda < \lambda_0$, where dissipation occurs. The value of this flux ε determines the local properties of the turbulence. This hypothesis is equivalent to the natural assumption that the energy transfer between modes is of a resonance character, in which, as we have seen, the energy of a mode of scale λ can be transferred only to modes with nearly the same scale. Thus a portion of energy handed down from large to smaller scales must pass through the entire range of scales of motion almost to λ_0. We may assert, therefore, that for each scale λ the value of ε is determined only by the fluctuation level at this scale, that is by the value of the spectral function for $k = 2\pi/\lambda$. In other words, ε must be expressible in terms of k and I_k only. The only dimensionally correct combination is

$$\varepsilon \sim \sqrt{I_k k^3}\, \rho I_k k \tag{III. 38}$$

since $I_k k^3$ has the dimension t^{-2}, and kI_k the dimension of a square of the velocity. From eqn. (III.38) the well known 5/3 law of Kolomogorov-Obukhov follows immediately:

$$I_k\, dk = C\left(\frac{\varepsilon}{\rho}\right)^{2/3} k^{-5/3}\, dk \tag{III. 39}$$

where C is a universal dimensionless constant.

This law agrees well with experimental data. It has recently been shown experimentally that ε is itself a random quantity undergoing large fluctuations, and in accordance with this Kolomogorov and Obukhov have made some changes in the theory, but we need not go into details of these in this review.

Thus the 5/3 law can be obtained from very general considerations of dimensional theory and the hypothesis of the absence of strong interaction

between pulsations of very different scales. To obtain the spectrum over the whole range of wave-numbers, including the viscous region $\lambda-\lambda_0$, more precise ideas about the character of the interaction between the pulsations are required. So far a large number of various hypotheses of a phenomenological character has been proposed to represent this interaction. The spectra which can be obtained from these hypotheses are very similar to each other (see ref. (62)), and in the inertial range all phenomenological theories, except those which are obviously wrong, lead to the 5/3 law.

(c) *Wind Waves*

As an example very similar to phenomena which occur in a plasma, let us discuss the development and interaction of waves on a high sea. The dispersion relation for gravitational waves is defined by the relation

$$\omega = \sqrt{gk} \tag{III. 40}$$

where g is the gravitational acceleration.

This dispersion relation is of non-decay type. This means that for small amplitude of the oscillations the principal part must be played by four wave processes involving the conversion of two waves into two other waves. Using perturbation theory we can obtain a kinetic wave equation for this type of interaction, but this equation is very complex and its solution is difficult. A much simpler method is the phenomenological approach in which the general form of the spectral function is established from physical considerations (refs. 70, 71).

Suppose that at some initial moment a wind starts to blow with a constant velocity above the surface of the sea. At first, when the surface of the sea is almost unperturbed, the principal mechanism governing the development of the waves will be their build-up by resonance with turbulent pulsations of the atmospheric pressure. The energy of each mode increases linearly with time. But as soon as the slope of the wavy surface attains some finite value, an instability mechanism will begin to play the principal part. The physical origin of this instability can be explained by reference to Fig. 14. When the

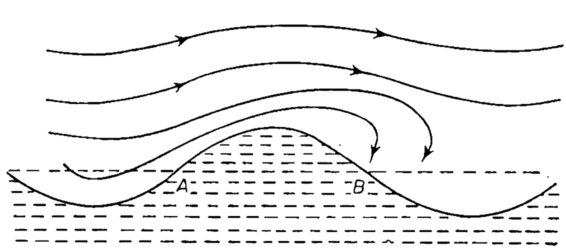

FIG. 14. Excitation of waves on water surface

slope of the wave is large enough, the air pressure will be slightly larger on the side A facing the wind than on the side B screened from the wind. This difference in pressure, due in the last analysis to the eddying of the wind above the region B, leads to an effective resonant transfer of energy from the

wind to the wave, since where the surface is rising the pressure is slightly smaller, and where it is falling, it is slightly larger than the average pressure.

The instability mechanism, which is in fact non-linear, leads to an exponential increase of the amplitude of the wave. But since it operates only for a considerable slope, which for a given amplitude will be larger the smaller the wavelength, only perturbations with wavelength smaller than some critical wavelength will increase exponentially at any given time. In going towards longer wavelength the spectral function must therefore decrease very rapidly with λ, probably more or less exponentially.

In the short wave region of the spectrum the amplitude in the steady state is determined by the non-linear interaction between the modes. The simplest sufficiently strong interaction between the waves is their collapse, as a result of which the " white crests " appear at the surface of the sea. This interaction limits the amplitude of the oscillations to such an extent that pointed " tips " appear on the waves. These " tips " correspond to discontinuities of the gradient so that the second derivative of the surface displacement ξ must contain δ-functions, and for large k values the Fourier transform of ξ must decrease as $\xi \sim k^{-2}$. It follows, therefore, that the spectral function of the vertical displacement ξ for large k values in the equilibrium region must behave as $I_k dk \sim k^{-4} d\mathbf{k}$ where $d\mathbf{k} = dk_x dk_y \sim k dk$. Transforming from the variable k to ω we obtain $I_\omega d\omega \sim d\omega/\omega^5$. This spectrum has been obtained by Phillips (70). It agrees well with experimental data relating to the short wave region of the spectrum. The total spectrum of the oscillations of the surface, measured experimentally, can be represented satisfactorily by the function

$$I_\omega = \alpha g^2 \omega^{-5} e^{-\omega_0{}^2/\omega^2} \tag{III. 41}$$

where α and ω_0 depend only on the time. This spectrum agrees perfectly with the above physical argument.

IV. TURBULENCE IN A PLASMA

1. Non-linear Damping of Langmuir Waves

THE discussion of specific turbulent processes in a plasma is best begun with the simplest case, that of Langmuir oscillations. In particular we shall consider here in some detail the non-linear damping of waves due to the resonant interaction. Assuming that the mean wave number of the wave packet under consideration is appreciably smaller than the reciprocal Debye radius D^{-1} we can neglect the linear Landau damping and retain only the second and third terms in the kinetic wave eqn. (II.51) (the last term disappears since Langmuir oscillations are of the non-decay type).

In the matrix element

$$v_{\mathbf{k}\omega,\mathbf{k}'\omega'} = -\frac{4\pi e}{k^2}\int\{(\mathbf{g}_{\mathbf{k}\omega}\mathbf{k}')(\mathbf{g}_{\mathbf{k}''\omega''}\mathbf{k}'')f+(\mathbf{g}_{\mathbf{k}\omega}\mathbf{k}'')(\mathbf{g}_{\mathbf{k}'\omega'}\mathbf{k}')f\}\,d\mathbf{v} \quad \text{(IV. 1)}$$

we keep only the first term, since for a small frequency difference $\omega'' = \omega-\omega'$ the quantity $\mathbf{g}_{\mathbf{k}''\omega''} \gg \mathbf{g}_{\mathbf{k}\omega}, \mathbf{g}_{\mathbf{k}'\omega'}$. By integrating by parts we replace $\mathbf{g}_{\mathbf{k}\omega}\mathbf{k}'$ by $-\frac{e}{m}\frac{\mathbf{k}\mathbf{k}'}{(\omega-\mathbf{k}\mathbf{v})^2} \cong -\frac{e}{m}\frac{\mathbf{k}\mathbf{k}'}{\omega^2}$, and the remaining integral can be expressed in terms of the dielectric permeability (II.43). Thus, in zero approximation

$$v^{(0)}_{\mathbf{k}\omega,\mathbf{k}''\omega''} \cong \frac{e}{m}\frac{(\mathbf{k}\mathbf{k}')}{\omega^2}\frac{k''^2}{k^2}[\varepsilon(\mathbf{k}'',\omega'')-1] \quad \text{(IV. 2)}$$

In this expression unity can be neglected compared with ε, since for $\omega < kv_e$ $\varepsilon(\mathbf{k}\omega) \cong \frac{m\omega_0^2}{k^2 T_e} \gg 1$. Thus $\varepsilon(\mathbf{k}''\omega'')$ cancels in the third term in (II.43). Replacing $\mathbf{k}'\mathbf{g}_{\mathbf{k}\omega}$ by its approximate value $-\frac{e}{m}\frac{\mathbf{k}\mathbf{k}'}{(\omega-\mathbf{k}\mathbf{v})^2} \cong -\frac{e}{m}\frac{\mathbf{k}\mathbf{k}'}{\omega^2}$ in the first integral, it is easily seen that this integral is proportional to $v_{\mathbf{k}''\omega'',\mathbf{k}\omega}$, and the two integral terms compensate one another exactly. This means that we must consider the thermal corrections which we initially neglected.

To determine which small corrections we must consider in particular, let us examine the matrix element

$$v_{\mathbf{k}''\omega'',\mathbf{k}\omega} = \frac{4\pi e}{k''^2}\int\{(\mathbf{g}_{\mathbf{k}''\omega''}\mathbf{k}')(\mathbf{g}_{\mathbf{k}\omega}\mathbf{k})f+(\mathbf{g}_{\mathbf{k}''\omega''}\mathbf{k})(\mathbf{g}_{-\mathbf{k}'-\omega'}\mathbf{k}')f\}\,d\mathbf{v} \quad \text{(IV. 3)}$$

The main contribution to this integral is supplied by the region where $\mathbf{g}_{\mathbf{k}''\omega''}f \to \infty$, i.e. where $\mathbf{g}_{\mathbf{k}\omega} \approx \mathbf{g}_{\mathbf{k}'\omega'}$. Therefore the term with the second

derivative of f with respect to the velocity, arising from the operation of $\mathbf{g}_{\mathbf{k}''\omega''}$ on f, is very small. Neglecting this term we can assume that the operator $\mathbf{g}_{\mathbf{k}''\omega''}$ acts only on $\mathbf{g}_{\mathbf{k}\omega}$ and $\mathbf{g}_{-\mathbf{k}',\,-\omega'}$. Performing the differentiation and keeping only the largest term, we have

$$v^{(0)}_{\mathbf{k}''\omega'',\mathbf{k}\omega} \cong \frac{4\pi e}{k''^2}\frac{e}{m}\frac{\mathbf{k}\mathbf{k}'}{\omega^2}\int (\mathbf{g}_{\mathbf{k}''\omega''}\mathbf{k}'')f\,d\mathbf{v} = \frac{e}{m}\frac{(\mathbf{k}\mathbf{k}')}{\omega^2}\{\varepsilon(\mathbf{k}'',\omega'')-1\} \qquad \text{(IV. 4)}$$

This shows that it is pointless to consider small real correction terms in the matrix element $v_{\mathbf{k}\omega,\,\mathbf{k}''\omega''}$, since the corresponding contribution will be proportional to $1-\varepsilon^{-1}(\mathbf{k}''\omega'')$, and its imaginary part of the order of $|\varepsilon(\mathbf{k}''\omega'')|^{-2} \ll 1$ and thus extremely small. It is sufficient to consider only the small terms of the first and second order of smallness with respect to kv/ω in the operators $\mathbf{g}_{\mathbf{k}\omega}$ in the first integral term in eqn. (II.43) and in the matrix element $v_{\mathbf{k}\omega,\,\mathbf{k}''\omega''}$. Collecting all these small terms and writing $\dfrac{\partial\varepsilon'}{\partial\omega} \approx \dfrac{2}{\omega_0}$, we finally obtain

$$\frac{\partial I_{\mathbf{k}}}{\partial t} = \frac{I_{\mathbf{k}}\omega_0^4}{2mk^2n^2\omega^5}\int (\mathbf{k}\mathbf{k}')^2(\mathbf{k}\mathbf{v})^2\delta(\omega''-\mathbf{k}''\mathbf{v})\mathbf{k}''\frac{\partial f}{\partial v}d\mathbf{v}\,I_{\mathbf{k}'}\,d\mathbf{k}' \qquad \text{(IV. 5)}$$

Since for Langmuir waves $\omega'' \sim \omega_0(kD)^2 \ll \omega_0$, $\mathbf{k}\mathbf{v} = \mathbf{k}'\mathbf{v}+\omega'' \approx \mathbf{k}'\mathbf{v}$ and with an accuracy up to higher order than the second in kv/ω $(\mathbf{k}\mathbf{v})^2$ can be replaced by $(\mathbf{k}\mathbf{v})$ $(\mathbf{k}'\mathbf{v})$. In this approximation the factor multiplying $I_{\mathbf{k}}$ inside the integral in the expression (IV.5) is anti-symmetric under the interchange of $\mathbf{k}$ and $\mathbf{k}'$. It follows, therefore, that in this approximation Langmuir waves are scattered without energy loss (9)

$$\frac{\partial}{\partial t}\int k^2 I_{\mathbf{k}}\,d\mathbf{k} = 0$$

According to (IV.5) the characteristic reciprocal diffusion time of the waves in $\mathbf{k}$-space is of the order

$$\nu \sim \frac{k_0^3 v_e^3}{\omega_0^2}\frac{1}{nT}\int k^2 I_{\mathbf{k}}\,d\mathbf{k} \qquad \text{(IV. 6)}$$

where k_0 is the mean value of the wave number. The damping of the waves is described by small terms, not considered here, and the corresponding damping decrement is of the order of $(kD)^2\nu$.

The energy damping rate can be determined from the following considerations. By integration by parts it can easily be shown that

$$v_{\mathbf{k}\omega,\,\mathbf{k}''\omega''} = \frac{k''^2}{k^2}v_{\mathbf{k}''\omega'',\,\mathbf{k}\omega} \qquad \text{(IV. 7)}$$

and we can see from the expression (IV.3) for the matrix element $v_{\mathbf{k}''\omega'',\,\mathbf{k}\omega}$ that if the exponentially small remainders at the points $\omega = \mathbf{k}\mathbf{v}$ and $\omega' = \mathbf{k}'\mathbf{v}$ are neglected, and only the remainder at the point $\omega'' = \mathbf{k}''\mathbf{v}$ retained, the exchange of $\mathbf{k}$ and $\mathbf{k}'$ is equivalent to taking the complex conjugate. In other words, the matrix element satisfies the symmetry condition

$$v_{\mathbf{k}''\omega'',\,\mathbf{k}\omega} = v^*_{-\mathbf{k}''-\omega'',\,\mathbf{k}'\omega'} \qquad \text{(IV. 8)}$$

From this relation and relation (IV.7) it can easily be seen that the imaginary part of the coefficient of $I_{\mathbf{k}'}$, in the expression below the integral in the third term on the right hand side of eqn. (II.51), is simply multiplied by $-k^2/k'^2$ when $\mathbf{k}$ and $\mathbf{k}'$ are interchanged. The second term in eqn. (II.51) has a similar symmetry. Making use of this symmetry we find that in the absence of decay the non-linear terms in eqn. (II.51) conserve the " wave number " $N_{\mathbf{k}} = \mathscr{E}_{\mathbf{k}}/\omega_{\mathbf{k}} = \frac{k^2}{8\pi} I_{\mathbf{k}} \left|\frac{\partial \varepsilon'}{\partial \omega}\right|$. In other words, during the scattering of Langmuir waves on the electrons, the wave energy decreases in proportion to the frequency

$$\mathscr{E}_{\mathbf{k}}/\omega_{\mathbf{k}} = \text{const, i.e. } \mathscr{E}_{\mathbf{k}} = \mathscr{E}_{\infty}\left(1 + \frac{3k^2T}{2m\omega_0^2}\right) \tag{IV. 9}$$

where the constant $\mathscr{E}_{\infty}$ represents the energy for $k \to 0$, i.e. $t \to \infty$. From eqn. (IV.9) the rate of damping of the energy can be determined from the rate of decrease of the wave number.

The result so obtained admits of an explicit quantum mechanical interpretation. This process represents a scattering of plasmons at electrons. The condition $\omega'' = \mathbf{k}''\mathbf{v}$ required by the δ-function represents in fact only conservation of energy:

$$\hbar\omega_{\mathbf{k}} - \hbar\omega_{\mathbf{k}'} + \Delta\mathbf{p}\mathbf{v} = 0$$

where $\Delta\mathbf{p} = -h\mathbf{k}''$ is the momentum transferred to the electron. When the electron energy states are populated normally, with $\frac{\partial f}{\partial v} < 0$, each plasmon loses energy on the average during scattering, so that its frequency and hence its wave number decrease. Since the frequency of a Langmuir plasmon depends only very weakly on the momentum k, the variation of its energy during scattering will be considerably smaller (by a factor k^2D^2) than the variation of momentum. For an inverted population of the electron states (i.e. for a " negative " temperature) scattering is associated with an increase in the energy of the plasmon (74).

For small amplitude of oscillation where higher wave interaction processes can be neglected, the scattering of the waves at the particles considered above is the main process. In contrast to conventional hydrodynamic turbulence where the interaction of the modes leads to a cascade of the energy towards large k, in the case of Langmuir oscillations the energy flux is directed towards the region of small k, where the linear damping is exponentially small.

It can be seen from (IV.9) that for small values of k_0D the energy of the Langmuir oscillations changes only slightly for $t \to \infty$. In other words, single scattering processes of Langmuir waves on the electrons do not lead to their total relaxation in a homogeneous plasma. But when we consider an inhomogeneous plasma, we must add terms of the form $\mathbf{U}_{\mathbf{k}}\frac{\partial I_{\mathbf{k}}}{\partial \mathbf{r}}$ and $\frac{\partial \omega_{\mathbf{k}}}{\partial \mathbf{r}}\frac{\partial I_{\mathbf{k}}}{\partial \mathbf{k}}$

to the kinetic equation for the waves. The second term, which is the more important, describes the energy flux in k-space due to the distortion of the wave packets. At small k, the corresponding rate of change in the mean wave vector associated with this flux $\dot{k}_0 = \dfrac{d\omega_0}{dx} = \kappa\omega_0$ becomes comparable to νk_0 and k_0 ceases to vary with time. Using eqn. (IV.6) we can write the condition for this in the following form $\dfrac{\nu k_0 D}{\omega_0} = (k_0 D)^4 \dfrac{\mathscr{E}}{nT} = \kappa D$. From this we find that $\dot{\mathscr{E}} = -\nu (k_0 D)^2 \mathscr{E} = -\kappa k D^2 \omega_0 \mathscr{E}$. Since the equilibrium value $k_0 D = \left(\kappa D \dfrac{nT}{\mathscr{E}}\right)^{\frac{1}{4}}$ depends only weakly on $\mathscr{E}$ and κ, the damping time of the Langmuir waves in an inhomogeneous plasma is to order of magnitude given by $(D\kappa\omega_0)^{-1}$. The time necessary for a deformation of the wave packet such that the wave number k attains the value D^{-1} will be of the same order of magnitude; at this wave number Landau damping becomes important. In a homogeneous plasma, where this mechanism does not operate, the relaxation of the Langmuir waves is determined by the slower 4-wave interaction processes and by the scattering of the waves at the ion fluctuations.

2. Excitation of Ion Oscillations by an Electron Current

(a) *Excitation of Ion-sound Waves*

Let us consider the problem of the excitation of ion-acoustic waves by an electron current (72). Suppose that the ion temperature T_i is considerably smaller than the electron temperature T_e. Such a situation arises for instance in a weakly ionised plasma where the ions lose energy in collisions with cold neutral gas atoms, or in non-stationary conditions where the ions are not heated by the electrons. The results of the stationary problem considered here can be used for the non-stationary case, since the time necessary for the establishment of the stationary oscillation spectrum is relatively small.

For $T_i \ll T_e$ ion oscillations are excited when the directed (current) velocity of the electrons u exceeds the velocity of sound $c_s = \sqrt{\dfrac{T_e}{m_i}}$ (see (75)–(77)). As u increases above c_s the excitation of oblique waves becomes possible as well as of those propagating parallel to the current, and for $u \gg c_s$ the cone of permissible directions of the unstable waves opens up, so that practically all waves having a positive projection of the phase velocity in the direction of u are unstable.

Two waves with wave numbers of about the same size and propagating at an angle to one another set up beats with small phase velocity since in this case $\omega'' = \omega - \omega' \approx 0$ and $k'' \neq 0$. Since the decay of ion-acoustic waves is prohibited, the non-linear Landau damping at the beats makes the main contribution to the non-linear interaction of the waves. This damping limits

the amplitude of the waves, and in stationary conditions some oscillation spectrum independent of time must be established.

The non-linear damping of ion-acoustic waves on the ions resembles the damping of Langmuir waves on the electrons. The considerations of the preceding paragraph can therefore be largely repeated. We can again omit the last term on the right hand side of the kinetic wave eqn. (II.51), because decay is prohibited. Moreover, the non-linear damping of the waves on the electrons can again be neglected because it will be described by the same terms as for Langmuir oscillations, except that m_e is replaced by m_i. As in the case of the electron oscillations, the non-linear terms in zero approximation with respect to the small parameter $kv/\omega \sim \sqrt{\dfrac{T_i}{T_e}}$ cancel each other out. The first order correction, which is in fact of the same order of magnitude, also vanishes and the next two terms of the expansion have to be considered.

In zero approximation the matrix elements are determined only by the ions and are given by the following relation:

$$v^{(0)}_{\mathbf{k}\omega,\mathbf{k}''\omega''} = \frac{k''^2}{k^2} v^{(0)}_{\mathbf{k}''\omega'',\mathbf{k}\omega} = -\frac{4\pi e^3(\mathbf{kk}')}{m_i^2 k^2 \omega^2}\left(1+\frac{2\mathbf{kk}''\omega''}{k''^2\omega}\right)\int \mathbf{k}''\frac{\partial f_i}{\partial \mathbf{v}}(\omega''-\mathbf{k}''\mathbf{v})^{-1}\,d\mathbf{v} \tag{IV. 10}$$

i.e. they are proportional to the integral $\int \mathbf{k}''\dfrac{\partial f_i}{\partial \mathbf{v}}(\omega''-\mathbf{k}''\mathbf{v})^{-1}\,d\mathbf{v}$.

The dielectric permeability also contains this integral

$$\varepsilon(\mathbf{k}'',\omega'') = 1+\frac{4\pi e^2}{m_i k''^2}\int \mathbf{k}''\frac{\partial f_i}{\partial \mathbf{v}}(\omega''-\mathbf{k}''\mathbf{v})^{-1}\,d\mathbf{v}+\frac{4\pi e^2 n}{k''^2 T_e} \tag{IV. 11}$$

In this case the integral term in (IV.11) is considerably (of the order of T_e/T_i times) larger than unity, which may be neglected. Thus in the second integral term of the wave eqn. (II.50) it is sufficient to consider the corrections of the second and third orders of smallness in the imaginary parts of the matrix elements only:

$$\operatorname{Im} v_{\mathbf{k}\omega,\mathbf{k}''\omega''} = \frac{k''^2}{k^2}\operatorname{Im} v_{\mathbf{k}''\omega'',\mathbf{k}\omega} = \int \frac{\mathbf{kk}'}{(\omega-\mathbf{kv})^2}\pi\delta(\omega''-\mathbf{k}''\mathbf{v})\mathbf{k}''\frac{\partial f_i}{\partial \mathbf{v}}\,d\mathbf{v} \tag{IV. 12}$$

These corrections can be obtained without difficulty and the kinetic equation for ion-acoustic waves reduces to the following form:

$$\frac{1}{2}\frac{\partial \varepsilon'}{\partial \omega}\frac{\partial I_{\mathbf{k}}}{\partial t}+\operatorname{Im}\varepsilon I_{\mathbf{k}} = \frac{\Omega_0^4 I_{\mathbf{k}}}{m_i\omega^6 k^2 n^2}\int (\mathbf{kk}')^2\left\{\left(1-3\frac{\mathbf{kk}''\omega''}{k''^2\omega}\right)(\mathbf{kv})^2+\right.$$
$$\left.+\left(3-4\frac{\mathbf{kk}''\omega''}{k''^2\omega}\right)\frac{(\mathbf{kv})^3}{\omega}-\frac{(\mathbf{kk}'')^2}{k''^4}\omega''^2\right\}\delta(\omega''-\mathbf{k}''\mathbf{v})\mathbf{k}''\frac{\partial f_i}{d\mathbf{v}}I_{\mathbf{k}'}\,d\mathbf{v}\,d\mathbf{k}' \tag{IV. 13}$$

where $\Omega_0^2 = \dfrac{4\pi e^2 n}{m_i}$, I_k is the spectrum of the potential and $\varepsilon' = \operatorname{Re}\varepsilon$.

For $T_i/T_e \to 0$ when the main contribution to the integral comes from the region $\omega'' \approx 0$, this equation takes the form

$$\frac{\Omega_0^2}{\omega^3}\frac{\partial I_{\mathbf{k}}}{\partial t} + \operatorname{Im} \varepsilon I_{\mathbf{k}} = \frac{\Omega_0^4 T_i I_{\mathbf{k}}}{m_i^2 \omega^3 \omega'^3 k^2 n} \int (\mathbf{kk}')^2 \{[\mathbf{kk}'\mathbf{k}''^2 - (\mathbf{kk}'')(\mathbf{k}'\mathbf{k}'')]\delta'(\omega'')\} I_{\mathbf{k}'}\, d\mathbf{k}' \tag{IV. 14}$$

where δ' denotes the derivative of the δ-function and $\omega'' = c_s|k-k'|$.

For $T_i \ll T_e$ the linear Landau damping on the ions can be neglected and there remains in Im ε only the contribution due to the electrons which leads to the build-up of the oscillations. Including the wave damping due to collisions between ions and neutral gas atoms, and supposing that the electrons have a Maxwellian velocity distribution displaced by a velocity u along the z axis, we obtain the following expression for Im ε:

$$\begin{aligned} -k^2 \operatorname{Im} \varepsilon &= \frac{4\pi^2 e^2}{m} \int \mathbf{k} \frac{\partial f_e}{\partial \mathbf{v}} \delta(\omega - \mathbf{kv})\, d\mathbf{v} - \frac{\Omega_0^2}{4c_s^2} \frac{1}{\omega\tau_i} \\ &= \frac{\omega_0^2}{\sqrt{2} v_e^3} (u \cos\theta - c_s) - \frac{\Omega_0^2}{4c_s^2} \frac{1}{\omega\tau_i} \end{aligned} \tag{IV. 15}$$

where $v_e^2 = \dfrac{2T_e}{m_e}$, $\omega_0^2 = \dfrac{4\pi e^2 n}{m}$, $\dfrac{1}{\tau_i}$ is the mean collision frequency of the ions with the neutral gas atoms, and $\cos\theta = k_z/k$. Thus the collisional damping decreases as ω increases. For a sufficiently large τ_i it is appreciable only at very small ω and its effect may be included by simply cutting off the spectral function towards low frequencies at the frequency at which Imε changes its sign, so that the waves damp. In this approximation the quantity k^2Imε in the principal region of wave numbers can be considered independent of k. It follows immediately from (IV.14) that $I_k \sim 1/k^3$. We can demonstrate this in more detail. Suppose for simplicity that the waves are built up only in a relatively narrow cone of directions within an angle θ_0. (Such a situation arises, for instance, when the plasma is contained in a cylindrical tube which absorbs waves which propagate in the transverse direction.) In this case $k'' \sim \theta_0 k$, and consequently the quantity $\mathbf{kk}'' \sim k''^2$ can be neglected compared with kk''. Transforming to k-space and using spherical co-ordinates, and integrating the δ' term in (IV.14) by parts, we obtain

$$\frac{\Omega_0^2}{\omega^3}\frac{\partial I_{\mathbf{k}}}{\partial t} = -\operatorname{Im} \varepsilon I_{\mathbf{k}} + I_{\mathbf{k}} \frac{\Omega_0^4 m_i^2 T_i}{T_e^4 n} \left\{ \int k^2 \theta_s^2 \frac{\partial I_{k\Omega'}}{\partial k} d\Omega' + 3k \int \theta_s^2 I_{k\Omega'}\, d\Omega' \right\} \tag{IV. 16}$$

where θ_s is the angle between k and k' and $d\Omega'$ is the solid angle element.

Integrating this equation with respect to Ω assuming $<\theta_s>^2 = \theta^2$, and denoting by I_k the integral of $I_{\mathbf{k}}$ over Ω, we obtain

$$\frac{\partial I_k}{\partial t} = 2\gamma I_k + A I_k \left(2k^5 \frac{\partial I_k}{\partial k} + 6k^4 I_k \right) \tag{IV. 17}$$

where $\gamma = -\frac{\omega^3}{2\Omega_0^2}\langle \mathrm{Im}\,\varepsilon\rangle$ is the mean growth rate of the small perturbations, and $A = \frac{\Omega_0^2 T_i \theta_0^2}{2c_s T_e^2 n}$. This equation can be written in more explicit form:

$$\frac{\partial I_k}{\partial t} - \frac{1}{k^2}\frac{\partial}{\partial k}(Ak^7 I_k^2) = 2\gamma_k I_k - Ak^4 I_k^2 \tag{IV. 18}$$

Noting that the energy of the waves is proportional to I_k and is given by

$$\mathscr{E}_k = m_i n v_k^2 = \frac{e^2 n}{m_i T_e} I_k \tag{IV. 19}$$

we can interpret eqn. (IV.17) as the wave energy transfer equation. The second term on the left hand side describes the diffusive energy transfer towards the smaller wave numbers due to the scattering of waves on the ions, and the second term on the right hand side describes the non-linear damping of the waves at the ions. As we have established earlier, in the principal region of wave numbers $k^2 \mathrm{Im}\,\varepsilon = \mathrm{const}$, so that γ_k is proportional to k, $\gamma_k = \alpha k$ say. Thus, from eqn. (IV.17) in steady state conditions, when $\frac{\partial I_k}{\partial t} = 0$, $I_k = \frac{\alpha}{2Ak^3}\ln\frac{k_0}{k}$, where $k_0 = \mathrm{const}$. The value of k_0 can be determined from the condition that the spectral function vanishes at $k \sim D^{-1}$, since at larger values of k the growth rate becomes negative. From this we obtain $k_0 \sim D^{-1}$. Moreover, the spectrum $I_k \sim k^{-3}\ln\frac{1}{kD}$ must be cut off at small k, where $\mathrm{Im}\,\varepsilon$ changes its sign.

We note that the non-linear terms in eqn. (IV.18) do not have a " wave number " $N_k = \mathscr{E}_k|\omega_k|^{-1}$. In other words, according to this equation $\int N_k d\mathbf{k} = \mathrm{const}$ for $\gamma = 0$. This result follows immediately from eqn. (IV.14). Thus the non-linear interaction of the ion-sound waves constitutes a coherent scattering at the ions, during which the " wave number " is conserved and the energy decreases in proportion to the frequency.

We obtain the frequency dependence from the relation $I_k d\mathbf{k} = I_\omega d\omega$, which gives $I_\omega \sim \omega^{-1}\ln\frac{\Omega_0}{\omega}$. Thus the energy of ion oscillations in the steady state is concentrated in the region of low frequencies, that is near the lower limit of stability. On the other hand, the spectral function of the electric field $E_k^2 = k^2 I_k$, which determines the effect of the oscillations on the electron distribution function, reaches a maximum for $k \sim D^{-1}$.

Substituting for the growth rate in eqn. (IV.17) the value given by linear theory, we obtain the following expression for the spectral function I_k for $u \gg c_s$:

$$I_k \sim \frac{u}{v_e}\frac{T_e}{T_i}\frac{1}{7\theta_0^2}\frac{T_e^2}{4\pi e^2 k^3}\ln\frac{1}{kD}$$

This is a very large value, and it is obvious that in the presence of oscillations of this magnitude the distribution function of the longitudinal electron velocities cannot be Maxwellian. In other words, we need to consider the feed-back of the oscillations on the averaged distribution function. For this purpose it is sufficient to use the quasi-linear approximation.

In the presence of a sufficiently strong longitudinal magnetic field, such that $\Omega_e = \dfrac{eH}{m_e c} \gg \Omega_0$, the motion of the electrons across the magnetic field can be neglected, and the oscillations lead to the development of a plateau on the longitudinal velocity distribution function in the interval from c_s to u. As a result the growth rate is reduced by the factor $\dfrac{\tau_e}{v_e^2}\dfrac{e^2 E_{k_0}^2}{m^2}\dfrac{k_0}{c_s}$ where $k_0 \sim D^{-1}$ and τ_e is the mean collision time between an electron and the neutral gas atoms. Taking this effect into account the spectral function becomes to order of magnitude

$$I_k \sim \frac{1}{10\theta_0}\sqrt{\frac{u}{c_s}\frac{m_e}{m_i}\frac{T_e}{T_i}\frac{1}{\omega_0 \tau_e}} \ln \frac{1}{kD}\frac{T_e^2}{e^2 k^3} \tag{IV. 20}$$

so that even when the formation of the plateau is taken into account the oscillation amplitude is still fairly large.

In the absence of a magnetic field the effect of the oscillations on the electron distribution function becomes even more important. In this case each separate wave with wave vector **k** sets up a plateau in the **k** direction over a small region of velocity space of the form of a thin layer. This layer is perpendicular to **k** and situated at a distance c_s from the co-ordinate origin (see Fig. 15). Since $c_s \ll v_e$, the aggregate of all oscillations sets up a plateau

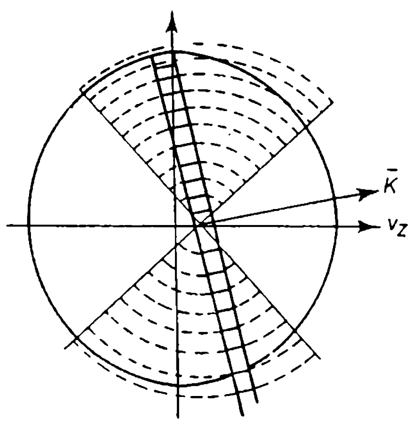

FIG. 15. Isotropisation of distribution function by ion acoustic oscillations

in the angular dependence of the distribution function in the resonance interaction region. Thus the ion acoustic oscillations developing in some cone of directions with aperture angle θ_0, must tend to make the electron distribution function isotropic in the external volume of a cone with aperture

angle $\pi/2-\theta_0$. For $u \gg c_s$, so that in an unbounded plasma practically all waves with a positive projection of the phase velocity on the z axis are unstable, the angle $\theta_0 \to \pi/2$, and the oscillations should lead to a complete " blocking " of the current, so that the directed velocity of the electrons could not exceed c_s. In a bounded plasma, let us say in the plasma of the positive column of a glow discharge, together with the effects just described, we should also take into account the energy transfer of the waves to the walls by adding the term $\mathbf{U}\dfrac{\partial I_{\mathbf{k}}}{\partial \mathbf{r}}$ to the kinetic wave equation. Accordingly waves propagating across the discharge should be damped, and the angle θ_0 may be fairly small even for $u \gg c_s$.

Let us note yet another circumstance. Above, we estimated the value of the spectral function averaged over the angles. If we wished to refine this calculation by determining the angular dependence, we would have to solve an integral equation with a degenerate core, and such an equation by no means always has a solution. In other words, the angular dependence of Im ε may turn out to be different from the angular dependence of the integral term in (IV. 16). The integral term then cannot completely compensate the growth rate, and there must be directions in $\mathbf{k}$ space in which the oscillation amplitude will continue to increase. We might expect, therefore, in the case of ion sound oscillations in an unbounded plasma, a tendency towards the formation of separate wave packets with fairly sharply defined angular dependence.

Moreover, from (IV.16) we see that the interaction of the waves increases as the angle θ_s between their wave vectors increases. Thus an individual mode may strongly interact with its " distant neighbours " but only weakly with waves travelling in the same direction as itself. Therefore the non-linear interaction not only does not resist, but even strengthens the tendency towards a localisation of the waves in a few separate directions. In an unbounded plasma this process will obviously develop until four-wave processes become important. In a bounded plasma we must add the term $\dfrac{\partial \omega_{\mathbf{k}}}{\partial \mathbf{r}} \dfrac{\partial I_{\mathbf{k}}}{\partial \mathbf{k}}$ into the kinetic equation for the short wave oscillations and this term will also affect their angular dependence. However, for the oscillations with a long wavelength of the order of the dimensions of the inhomogeneity, this approach is not suitable and in those cases where the condition for their build-up is fulfilled, we should expect a predominant development of one of these characteristic oscillations.

(b) *Build-up of Cyclotron Oscillations by the Electron Current*

The ion-wave instability considered above only occurs in a non-isothermal plasma where $T_e \gg T_i$. For $T_i \sim T_e$ ion acoustic waves do not occur because of the strong damping on the ions, and the growth of oscillations is possible only for $u \gtrsim v_e$ (77). However, in the presence of a magnetic field, which increases the " elasticity " of the plasma, oscillations can be excited at smaller values of the directed velocity u.

Let us first consider the case of a strong magnetic field where the magnetic field pressure is much larger than the plasma pressure, i.e. $\beta = \frac{8\pi p}{H^2} \ll 1$. Drummond and Rosenbluth (78) have shown that in this case the longitudinal electron flux may excite ion cyclotron oscillations. For $\beta \ll 1$ these oscillations are purely longitudinal, so that we can write $\mathbf{E} = -\nabla\varphi$. Their frequency is given by the relation (75):

$$\omega = \Omega_i \left\{ 1 + \frac{T_e}{T_i} e^{-s} I_1(s) + i\sqrt{\pi}\frac{T_e}{T_i} e^{-s} I_1(s) \frac{k_z u - \Omega_i}{k_z v_e} \right\} \tag{IV. 21}$$

where $\Omega_i = \frac{eH}{m_i c}$ is the ion cyclotron frequency, $s = k_\perp^2 \rho_i^2 = \frac{k_\perp^2 T_i}{m_i \Omega_i^2}$, $I_1(s)$ is the Bessel function of imaginary argument, and $v_e = \sqrt{\frac{2T_e}{m_e}}$.

Equation (IV.21) refers only to waves with small k_z, since the ion damping omitted in this equation can be neglected only for $k_z \lesssim \frac{1}{3}\frac{\omega - \Omega_i}{v_i}$. Since the quantity $\frac{\omega - \Omega_i}{\Omega_i} \approx \frac{T_e}{T_i} e^{-s} I_1(s)$ attains a maximum of $\sim 0{\cdot}2\frac{T_e}{T_i}$ for $s \approx 1{\cdot}5$,

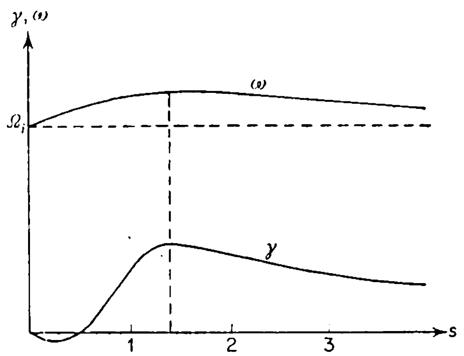

FIG. 16. Dependence of frequency and growth rate of cyclotron oscillations on wave number

we have $k_z \lesssim \frac{1}{15}\frac{T_e}{T_i}\frac{\Omega_i}{v_i}$, and then according to eqn. (IV.21) the instability develops only for

$$u > u_c \approx 15\frac{T_i}{T_e} v_i = 15\left(\frac{T_i}{T_e}\right)^{\frac{3}{2}} \sqrt{\frac{m_e}{m_i}} v_e \tag{IV. 22}$$

For a hydrogen plasma this condition becomes $u \gtrsim \frac{v_e}{3}\left(\frac{T_i}{T_e}\right)^{\frac{3}{2}}$. The dependence of the frequency ω and the growth rate γ on the wave number $k_\perp$ is shown qualitatively in Fig. 16. This relationship between γ and $k_\perp$ applies when the instability condition (IV.22) is satisfied by a considerable margin. As u decreases the band of unstable wave numbers becomes narrower.

Within the framework of the quasi-linear approximation, the problem of the sustained oscillations has been studied in the paper quoted above (78). These authors evaluated the oscillation amplitude at which a plateau appears in the electron distribution function. However, in the presence of a longitudinal electric field the development of the oscillations does not stop at this stage, since the electric field tends to destroy this plateau, and the distribution function will always tend to relax towards a displaced Maxwellian distribution, and the oscillation amplitude will continue to grow until the non-linear processes enter into the picture.

Oscillations with a frequency slightly larger than the cyclotron frequency set up beats at frequencies close to zero and to double the cyclotron frequency. The oscillations with very low frequency transfer energy to the ions very effectively and it will be these in particular which will lead to the limitation of the oscillation amplitude. The problem of sustained cyclotron oscillations has been considered by Petviashvili (79) and Karpman (80).

Since the phase velocity of cyclotron waves parallel to the magnetic field $\omega/k_z \sim u$ is considerably larger than the thermal velocity of the ions, the non-linear damping of the waves, as in the preceding paragraph, will be described by a term containing the function $\delta(\omega-\omega')$ beneath the integral. When we integrate the δ function with respect to $k_\perp$ we obtain a factor of the form $\left|\frac{d\omega}{dk_\perp}\right|^{-1}$, which vanishes at the point where ω reaches its maximum as a function of $k_\perp$. Consequently, at this point the amplitude of oscillation in the steady state must vanish, even though according to Fig. 16 the growth rate is greatest at this point.

When we consider the formation of the plateau in the electron distribution function, we obtain from the non-linear damping the following estimate of the equilibrium spectral function

$$d\mathbf{k}_\perp \int I_\mathbf{k}\, dk_z = A\frac{T_e T_i}{e^2}\left(\frac{u^3}{v_e v_i^2 \Omega_e \tau_e}\right)^{\frac{1}{2}} \frac{d\mathbf{k}_\perp}{k_\perp^2} \tag{IV. 23}$$

where A is a numerical factor of the order unity, $\Omega_e = \frac{eH}{m_e c}$, $\frac{1}{\tau_e}$ is the electron collision frequency and I_k the spectral function of the electric potential. This expression refers to the case where u is much greater than the critical value u_c. It gives a fluctuation level considerably exceeding the thermal fluctuation density which in this region of wavelengths is of the order of magnitude of $\int I_\mathbf{k}\, d\mathbf{k} \sim \frac{T_e^2}{e^2}(n_0 \rho_i^3)^{-1}$,

(c) *Magnetic Sound Build-up*

The phase velocity of the oscillations under consideration is of the order of magnitude of u, and as we increase $\beta = 8\pi p/H^2$ so that the phase velocity approaches the Alfvén speed, they can no longer be considered longitudinal. In other words, in this case curl $\mathbf{E} \neq 0$, the oscillations

become much more complex, and the growth of longer wave magnetosonic oscillations becomes possible.

The dispersion relation for magnetosonic oscillations including Landau damping has been determined in (81, 82). Extending these results to include the case of a longitudinal current, we obtain for $\omega \ll \Omega_i$:

$$\omega^2 = c_A^2 k^2 \left\{1 + i\frac{\sqrt{\pi}}{v_e k_z}(k_z u - k c_A)\frac{T_e}{m_i c_A^2}\frac{k_\perp^2}{k^2}\right\} \tag{IV. 24}$$

This shows that instability occurs for $u > c_A$, but the growth rate is comparatively small, given by $\gamma/\omega \sim \beta\frac{u}{v_e}$.

Equation (IV.24) can be used approximately almost up to $\omega \sim \Omega_i$, corresponding to $k\rho_i \sim \sqrt{\beta}$. When the wave number increases above this value, the magnetosonic oscillations change fairly rapidly into purely electronic oscillations corresponding to the so-called " atmospheric Whistlers ". The ions do not participate in these oscillations which cannot therefore be excited by a longitudinal current. Also for shorter wavelengths we have the cyclotron branch, which we considered earlier, which for $\beta \ll 1$ connects with the ion acoustic or slow magnetosonic wave.

Thus the magnetosonic oscillations can be excited only for $k\rho_i < \sqrt{\beta}$. Their dispersion relation is of the non-decay type (see Chapter II.1(b)). It is true that they can excite Alfvén waves by resonant transfer, but since the latter are practically not absorbed by particles, this effect does not lead to a limitation of the oscillation amplitude. Consequently even in this case the amplitude of the steady state oscillations is determined by the non-linear Landau damping at the beat frequencies. The mean square of the amplitude of the steady state oscillations E_k^2 can be estimated from considerations of the energy balance. We compare the rate of energy transfer from the electrons to the ions with the rate of absorption of energy by the ions, and obtain approximately

$$\tilde{\gamma}E_k^2 \equiv \gamma\left(\frac{\tau_e}{v_e^2}\frac{e^2}{m_e^2}\frac{1}{uk}E_k^2\right)^{-1} E_k^2 = BE_k^4 \tag{IV. 25}$$

where E_k^2 is some mean value of the spectral function, $k \sim \rho_i^{-1}\sqrt{\beta}$, $\gamma \sim \omega\beta\frac{u}{v_e}$ is the linear growth rate and $\tilde{\gamma}$ the growth rate as reduced by the formation of the plateau in the electron distribution function. The right hand side of (IV.25) represents the non-linear damping of the waves. To estimate the value of B we observe that the amplitude of the low frequency beat oscillations is directly related to the non-linear terms in the equation of motion for the ions and can be estimated as $E_k' \sim \frac{m_i k}{e}\frac{c^2 E_k^2}{H^2}$. The energy of these oscillations is in order of magnitude equal to $m_i n\frac{c^2}{H^2}E_k'^2 = m_i^3 n\frac{c^6 k^2}{e^2 H^6}E_k^4$ and its

absorption rate can be estimated as $\beta c_A k \frac{c^2 k^2}{H^2 \Omega_i^2} E_k^2 . m_i n E_k^2 \frac{c^2}{H^2}$, where the factor β takes into account that the interaction leads to damping only for oscillations with frequencies sufficiently close that the phase velocity of the beats is of the order of the ion thermal velocity (cf. the factor T_i/T_e in eqn. (IV.16)). Comparing this result with eqn. (IV.25) and noting that the energy of the main oscillations is of the order of magnitude $\sim m_i n E_k^2 \frac{c^2}{H^2}$ we obtain approximately $B \sim v_i k \beta \frac{k^2 c^2}{\Omega_i^2 H^2}$. Substituting this value in the relation (IV.25) and setting $k \sim \rho_i^{-1} \sqrt{\beta}$, we finally obtain an approximate value of the intensity of the oscillations of the electric field

$$E_k^2 \sim \frac{H^2 v_i^2}{c^2} \frac{u}{\sqrt{c_A v_e}} \frac{1}{\sqrt{\Omega_e \tau_e}} \tag{IV. 26}$$

It is apparent that the kinetic energy of the steady state oscillations is broadly speaking $\sqrt{\Omega_e \tau_e}$ times smaller than the thermal energy, and is concentrated in oscillations with frequencies near the cyclotron frequency Ω_i.

Both magnetosonic and cyclotron oscillations lead to a plateau of width u in the electron distribution function. As a result the electron current decreases by a small fraction of order u/v_e. This decrease of the current for a given electric field can be represented as an additional " anomalous " resistivity. Consequently the effective conductivity of the plasma σ is defined by the relation

$$\frac{1}{\sigma} \approx \frac{1}{\sigma_0}\left(1 + \frac{u}{v_e}\right) \tag{IV. 27}$$

where σ_0 is the conductivity due to binary collisions.

It can be shown that one-third of the energy dissipated by the additional resistivity is transferred to the oscillations and so to the ions. As a result additional heating of the ions occurs. In stationary conditions, when there are no energy losses from the ions, and the electron temperature is maintained at a given level, the anomalous heating of the ions must increase their temperature to such a value that the energy transferred to the ions $\frac{1}{3} \frac{u}{v_e} \frac{j^2}{\sigma_0}$ is equal to the energy $\frac{m_e}{m_i} \frac{n}{\tau_e} (T_i - T_e)$, transferred by the ions to the electrons. Writing $\sigma_0 = e^2 n \tau_e / m_e$, $j = enu$ we have

$$T_i \cong T_e \left(1 + \frac{1}{3} \frac{m_i}{m_e} \frac{u^3}{v_e^3}\right) \tag{IV. 28}$$

Thus for a sufficiently large value of u/v_e this turbulent heating of the ions may lead to a considerable " breakaway " of the ion temperature from that of the electrons.

In addition to this anomalous heating these oscillations must lead to an enhanced diffusion of the plasma across the magnetic field, since both the electrons and the ions participate in the oscillations. The diffusion is determined by the magnitude of the mean drift along the density gradient, parallel to the x axis, and the diffusion flow can be represented by

$$q = -\frac{d}{dx}\int \frac{c^2}{H^2}\pi E_{y\mathbf{k}}^2 \delta(\omega_{\mathbf{k}} - \mathbf{kv}) f_e \, d\mathbf{v}\, d\mathbf{k} \tag{IV. 29}$$

Substituting here the value given by eqn. (IV.26) for E_k^2 and considering that the only contribution to the integral comes from the resonance region, we obtain for the diffusion coefficient the approximate value

$$D_\perp \sim \frac{u^2}{v_e\sqrt{v_e c_a}} \cdot \frac{D_B}{\sqrt{\Omega_e \tau_e}} \tag{IV. 30}$$

where $D_B \sim \rho_i v_i$ is the Bohm value.

The cyclotron oscillations lead to a diffusion coefficient of the same order of magnitude.

3. Drift Instability of a Plasma

In this section we shall deal with the drift instability of a plasma which gives rise to anomalous diffusion of a rarified plasma in a homogeneous magnetic field. The drift instability was described by Tserkovnikov (19) who showed that in the presence of a temperature gradient in an inhomogeneous plasma oscillations may be excited with a phase velocity across the magnetic field of the order of the drift velocity of the particles. Such oscillations are naturally called drift waves.

In Tserkovnikov's paper (19) only waves which propagate across the magnetic field were considered. Later Rudakov and Sagdeev (20) considered the more general case of oblique drift waves which transform into ion acoustic waves as the angle between the wave vector and the magnetic field decreases. It has been shown in (19) that in the presence of a temperature gradient these may be growing waves.

A considerable influence on the further development of the theory of drift instability was the paper by Rosenbluth *et al.* (21) on the finite Larmor radius stabilization of the flute instability of a plasma. This paper stimulated a whole series of further investigations of the drift instability of the plasma with respect to short wave perturbations with transverse wavelength of the order of the mean Larmor radius of the ions.

These papers were concerned with the investigation of the stability of a plasma of such low density that particle collisions could be neglected. On the other hand, drift waves are also obtained in the investigation of the stability of a weakly ionised plasma where collisions between charged particles and neutral gas atoms are dominant. Timofeev (23) (see also (91, 92)) has shown that as the neutral gas pressure in a weakly ionised plasma is reduced, a peculiar instability appears which arises from a combination of the drift motion of the charged particles across the magnetic field and the diffusion

along the field. We shall call such an instability a drift-dissipative instability. It has been shown by Moiseev and Sagdeev (24) that such an instability also occurs in a dense fully-ionised plasma. It has also been shown in (85) and (92) that thermal conductivity and viscosity may appear in addition to diffusion as dissipative processes leading to instability.

In this section a brief survey will be given of the drift instabilities in an inhomogeneous plasma (a more comprehensive survey is due to Mikhailovskiĭ (22)). We shall assume that the plasma pressure is much smaller than the pressure of the magnetic field, i.e. $\beta = \frac{8\pi p}{H^2} \ll 1$. The magnetic field will be assumed homogeneous.

(a) *Drift Waves in an Inhomogeneous Plasma*

Consider a low pressure plasma in a homogeneous magnetic field H parallel to the z axis, and suppose that the plasma density varies along the x axis. For simplicity we shall assume that the plasma temperature is constant, and that the ions are cold. Then in equilibrium the ions are at rest, and the electrons drift with a velocity $v_0 = -\frac{T_e}{m_e \Omega_e n} \frac{dn}{dx}$ where $\Omega_e = \frac{eH}{m_e c}$. This is the so-called Larmor drift. We assume that $\frac{dn}{dx} < 0$ so that $v_0 > 0$.

We consider perturbations with transverse wavelength considerably smaller than the characteristic dimension $a = \kappa^{-1}$, where $\kappa = \frac{1}{n}\frac{dn}{dx}$. Such perturbations can be represented as plane waves of the form $\exp(-i\omega t + i\mathbf{kr})$. We limit the discussion to start with to the case in which $\omega \ll k_z c_A$, and we assume the electric field curl-free, so that we can write $\mathbf{E} = -\nabla\varphi$. Thus we are discussing first the longitudinal oscillations of the plasma. Provided k_z is not too small, so that the phase velocity of the wave is considerably smaller than the thermal velocity of the electrons, the latter follow a Boltzmann distribution, so that the perturbation of the electron density n_e can be expressed in terms of φ:

$$\frac{n_e}{n} = \frac{e\varphi}{T_e} \tag{IV. 31}$$

where n is the unperturbed density and T_e the electron temperature. The expression for the perturbation of the ion density n_i can be obtained from the continuity equation

$$\frac{\partial n_i}{\partial t} + \operatorname{div}(n\mathbf{v}_i) = 0 \tag{IV. 32}$$

where $\mathbf{v}_i$ is the macroscopic ion velocity, which can be determined from the equation of motion

$$\frac{\partial \mathbf{v}_i}{\partial t} = -\frac{e}{m_i}\nabla\varphi + \frac{e}{m_i c}[\mathbf{v}_i\mathbf{H}] \tag{IV. 33}$$

For $\omega \ll \Omega_i$ we then obtain the transverse and longitudinal components of the velocity

$$\mathbf{v}_{i\perp} = i\frac{c}{H}[\mathbf{hk}]\varphi + \frac{c}{H}\frac{\omega}{\Omega_i}\mathbf{k}_\perp\varphi \tag{IV. 34}$$

and

$$v_{iz} = \frac{e}{m_i}\frac{k_z}{\omega}\varphi \tag{IV. 35}$$

For $\omega \ll \Omega_i$ the second term in (IV.34) can be neglected and then the velocity of the ions is determined entirely by the drift in the electric field. In this approximation, using relations (IV.33), (IV.34) and (IV.35), we obtain

$$\frac{n_i}{n} = \left(\frac{\omega_*}{\omega} - \frac{k_z^2 T_e}{m_i\omega^2}\right)\frac{e\varphi}{T_e} \tag{IV. 36}$$

where

$$\omega_* = k_y v_0 = \frac{T_e k_y \kappa}{m_i \Omega_i}$$

In a dense plasma, where the Debye radius is negligibly small, all oscillations can be considered quasi-neutral, i.e. $n_i = n_e$. Substituting in this relation the values for the perturbations of the electron and ion densities obtained above, we arrive at the dispersion equation

$$\omega^2 - \omega\omega_* - k_z^2 c_s^2 = 0 \tag{IV. 37}$$

where $c_s = \sqrt{\frac{T_e}{m_i}}$ is the velocity of sound. This is in fact the required dispersion relation for drift waves. The dependence of ω on k_z determined by this equation is shown in Fig. 77. Curve (1) refers to a wave propagating

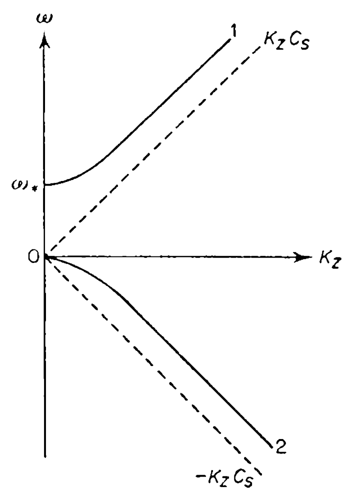

FIG. 17. Dispersion relation between ω and k_z for longitudinal oscillations of an inhomogeneous plasma

in the direction of the electron drift, for which $\omega/k_y > 0$, and Curve (2) to the wave propagating in the opposite direction, i.e. with $\omega/k_y < 0$. As we can see, the absolute values of the frequencies of these two waves differ slightly from one another. In particular, for $k_z \to 0$ the frequency of the first wave tends towards ω_*, while the frequency of the second tends towards zero.

The transverse phase velocity of the first wave for $k_z = 0$ is equal to the drift velocity of the electrons. In order of magnitude it is equal to $c_s r_H/a$, where $r_H = \sqrt{\dfrac{T_e}{m_i \Omega_i^2}}$ is the Larmor radius of the ions at the electron temperature (for the usual Larmor radius of the ions $\sqrt{\dfrac{T_i}{m_i \Omega_i^2}}$ we retain the notation ρ_i). For $k_z/k_y > r_H/a$ both drift waves go over into ion acoustic waves. Therefore it is natural to call the first the accelerated, and the second the decelerated ion acoustic wave. In a strong magnetic field $r_H \ll a$, and in this case the transition to a simple sound wave takes place for almost transverse propagation where $k_z/k_y \ll 1$.

Returning once more to the derivation of the dispersion equation, it is easy to see that the effect of the inhomogeneity on the oscillation frequency arises from the transverse drift of the ions. By itself this motion is incompressible, i.e. div $\mathbf{v}_{i\perp} = 0$, and in a homogeneous plasma it does not lead to a change in density. However, in an inhomogeneous plasma, even an incompressible displacement of the plasma ξ along the density gradient leads to a perturbation of the density $n_i = -\xi \dfrac{dn}{dx}$. This perturbation of the density leads to the change of the dispersion equation.

These considerations remain valid when the ion temperature is different from zero. For if the transverse wavelength is considerably longer than the Larmor radius of the ions ρ_i, and the phase velocity parallel to the z axis is considerably larger than the thermal velocity v_i, we can still use the hydrodynamic equation of motion (IV.33) only including the ion pressure gradient. This term leads to the Larmor drift $\dfrac{c}{H}[\mathbf{h}\nabla p_i]$, which for $H = \text{const}$ is incompressible and therefore supplies no contribution to the change in density. Thus, for $T_i \neq 0$ the dispersion relation (IV.37) is unchanged provided $\omega/k_z > c_s$. It follows, therefore, that even for $T_i = T_e$, so that $c_s \sim v_i$, and ion acoustic waves do not occur in a homogeneous plasma, a drift wave (the accelerated sound wave) may propagate in an inhomogeneous plasma with a frequency $\omega \approx \omega_*$. The phase velocity of this wave parallel to the magnetic field may considerably exceed the thermal velocity of the ions, and consequently it is not subject to strong ion Landau damping.

As k_z decreases the phase velocity of the accelerated wave increases and when k_z is sufficiently small it may reach either c_A or v_e, whichever is smaller. We limit our consideration here to the case $c_A \ll v_e$, i.e. $\beta = 8\pi p/H^2 \gg m_e/m_i$.

For $\frac{\omega}{k_z} \sim c_A$ the electric field can no longer be considered curl-free. For $\frac{\omega}{k_z} \sim c_A$ the lines of force of the magnetic field may be said to be no longer completely rigid and they become slightly bent. In this case it is possible purely formally to introduce a longitudinal potential ψ, determined by the relation $E_z = -\frac{\partial \psi}{\partial z}$, while since in the case of the slow oscillations under consideration the transverse component of the electric field can be assumed curl-free, we can retain the transverse potential φ; $\mathbf{E}_\perp = -\nabla_\perp \varphi$. By using these two potentials we take account of the bending of the lines of force, but continue to neglect any changes in field strength due to compression of the field. For $\psi = \varphi$ of course even the curvature of the field disappears.

Since according to our assumption the thermal velocity of the electrons considerably exceeds the phase velocity of the wave $\omega/k_z \sim c_A$, the electrons reach equilibrium along the lines of force leading to the following relation

$$T_e i k_z n_e + T_e \frac{H'_x}{H} \frac{dn}{dx} = i k_z e n \psi \tag{IV. 38}$$

where H'_x is the x-component of the perturbation in the magnetic field given by $H'_x = \frac{c}{\omega}[\mathbf{kE}]_x = i\frac{c}{\omega} k_y k_z(\varphi - \psi)$. Thus we obtain for the perturbation of the electron density

$$\frac{n_e}{n} = \frac{e\varphi}{T_e} + \frac{e}{T_e}(\psi - \varphi)\left(1 - \frac{\omega_*}{\omega}\right) \tag{IV. 39}$$

We need to relate φ and ψ. For this purpose we use the z component of the equation $\frac{\partial \mathbf{j}}{\partial t} = -\frac{c^2}{4\pi}\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{E}$, where j is the current density. For these oscillations we obtain

$$j_z = \frac{c^2}{4\pi\omega} k_z k_\perp^2 (\varphi - \psi) \tag{IV. 40}$$

To determine j_z, we use $\mathrm{div}\,\mathbf{j} = 0$, from which we obtain

$$j_z = i k_z^{-1}\,\mathrm{div}\,\mathbf{j}_\perp = i k_z^{-1} e\,\mathrm{div}\, n(\mathbf{v}_{i\perp} - \mathbf{v}_{e\perp})$$

Thus the longitudinal current arises from the small difference in the transverse velocities of the electrons and ions, so that in the expression (IV.34) for the ion velocity we must consider the second (inertial) term. In addition there is a small difference between the ion velocity and the electric drift velocity arising from the effect of the finite Larmor radius (21). In a strong magnetic field, the mean Larmor radius of the ions, although small, is nevertheless not negligible and the electric drift of the ions is not determined by the electric field at the centre of the Larmor circle, but by some value averaged over the Larmor circle. It is easily verified that when we average the electric field of

a plane wave over a circle of radius $v_\perp/\Omega_i$ and subsequently over the Maxwellian distribution in $v_\perp$, the effective electric field is less than the value at the centre of the circle by a factor $e^{-s} I_0(s)$, where $s = \frac{k_\perp^2 T_i}{m_i \Omega_i^2}$ and I_0 is the Bessel function of imaginary argument. For small s this factor is approximately equal to $1-s$, and in this case a more accurate expression for the ion velocity may be written down in the following form

$$\mathbf{v}_{i\perp} = \frac{ic}{H}[\mathbf{hk}]\varphi - \frac{ic}{H}[\mathbf{hk}]\frac{k_\perp^2 T_i}{m_i \Omega_i^2}\varphi + \frac{c}{H}\frac{\omega}{\Omega_i}\mathbf{k}_\perp \varphi \tag{IV. 41}$$

Substituting this expression for the ion velocity in div $\mathbf{j}_\perp$ and noting that $\mathbf{v}_{e\perp} = \frac{ic}{H}[\mathbf{hk}]\varphi$ we have

$$_z = -\frac{n m_i c^2}{H^2}\frac{k_\perp^2}{k_z}(\omega + \omega_*)\varphi \tag{IV. 42}$$

whence, using (IV.40), we obtain the following relation between φ and ψ

$$\psi - \varphi = -\varphi \frac{\omega(\omega + \omega_*)}{k_z^2 c_A^2} \tag{IV. 43}$$

Using this relation and (IV.31), we obtain the perturbation of the electron density:

$$\frac{n_e}{n} = \left(1 - \frac{\omega^2 - \omega_*^2}{c_A^2 k_z^2}\right)\frac{e\varphi}{T_e} \tag{IV. 44}$$

Finally, comparing this equation with the perturbation of the ion density (IV.36), neglecting the second term in brackets, we obtain the dispersion relation

$$(\omega - \omega_*)(\omega^2 + \omega_* \omega - c_A^2 k_z^2) = 0 \tag{IV. 45}$$

This splits up into two: $\omega = \omega_*$, which describes the drift wave which we already know, and

$$\omega^2 + \omega_* \omega - c_A^2 k_z^2 = 0 \tag{IV. 46}$$

which shows that for $k_z \to 0$ the Alfvén waves in an inhomogeneous plasma also go over into drift waves. To differentiate these waves from those considered earlier, we shall call one the accelerated Alfvén wave ($\omega/k_z > c_A$), the other the decelerated Alfvén wave ($\omega/k_z < c_A$).

Retaining small terms of the order $k^2\rho_i^2$ previously neglected, we obtain, for $\beta > \frac{m_e}{m_i}$, instead of an actual intersection of the branches, the picture shown in Fig. 18. There, as previously, waves with positive frequency (branches 1 and 3) propagate in the direction of the electron drift, and waves with negative frequency (branches 2 and 4) in the direction of the ion drift. In an inhomogeneous plasma, the usual sound and Alfvén waves are replaced by four different waves.

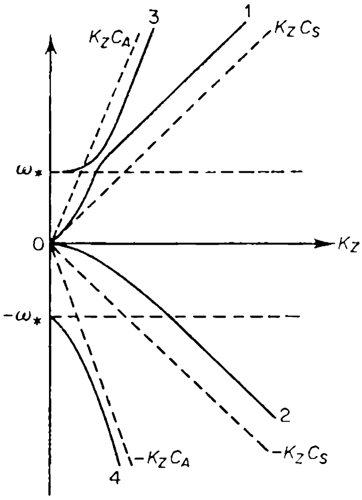

FIG. 18. Dependence of frequency of drift waves on wave number

In the following sections we investigate the stability of an inhomogeneous plasma to drift waves. In a collisionless plasma the instability arises from the interaction between the drift waves and resonant particles represented by Landau damping, which must be discussed on the basis of kinetic considerations. This calculation is carried out in Sections (b) and (c) where we consider separately long wave ($k_\perp \rho_i \ll 1$) and short wave ($k_\perp \rho_i \gtrsim 1$) perturbations. In Section (d) we consider the stabilization of the flute instability due to finite Larmor radius. In the following Section (e) the instability of a plasma with cold ions ($T_i = 0$) is considered in a weak magnetic field, when the Larmor radius of the ions calculated from the electron temperature exceeds the characteristic dimension of the inhomogeneity. In Section (f) the growth of cyclotron oscillations in an inhomogeneous plasma is considered. The following three sections are then devoted to the consideration of the drift-dissipative instability, related to collisions between particles, and in the last section non-linear drift oscillations will be discussed.

(b) *Drift Instability for* $k_\perp \rho_i \ll 1$

To describe the oscillations of a collisionless plasma with a wavelength much greater than the mean Larmor radius of the ions, the drift kinetic equation can be used (the same approximation has been used in ref. (20) which we follow in the present section). We shall limit the discussion to the case of longitudinal oscillations, and for these the linearised drift kinetic equation for particles with mass m and charge e has the following form

$$(-\omega + k_z v_z) f' - \frac{e}{m} k_z \varphi \frac{\partial f}{\partial v_z} - \frac{c}{H} k_y \varphi \frac{\partial f}{\partial x} = 0 \qquad \text{(IV. 47)}$$

where f is the unperturbed distribution function. With the aid of (IV.47) we obtain the density perturbation

$$n' = -e\varphi \int \left(\frac{k_z}{m}\frac{\partial f}{\partial v_z} + \frac{k_y}{m\Omega}\frac{\partial f}{\partial x}\right)\frac{d\mathbf{v}}{\omega - k_z v_z + i\nu} \tag{IV. 48}$$

where we have added the small positive quantity $\nu \to 0$ to pass the pole correctly. In the special case that the distribution function f is Maxwellian and the temperature constant, the density perturbation n' can be expressed in terms of the function

$$Y(z) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} \frac{e^{-x^2}\,dx}{z - x + i\nu} = 2e^{-z^2}\int_0^z e^{t^2}\,dt - i\sqrt{\pi}\,e^{-z^2} \tag{IV. 49}$$

Also $\dfrac{\partial f}{\partial v_z} = -\dfrac{mv_z}{T}f$, $\dfrac{\partial f}{\partial x} = -\kappa f$, so that we obtain from (IV.48)

$$\frac{n'}{n} = \left\{\frac{\omega + \omega_*}{k_z v_T} Y\left(\frac{\omega}{k_z v_T}\right) - 1\right\}\frac{e\varphi}{T} \tag{IV. 50}$$

where v_T is the thermal velocity. Using the quasi-neutrality condition we obtain the following dispersion equation

$$\frac{\omega + \omega_*}{k_z v_i} Y\left(\frac{\omega}{k_z v_i}\right) + \frac{\omega - \omega_*}{k_z v_e} Y\left(\frac{\omega}{k_z v_e}\right) = 2 \tag{IV. 51}$$

where

$$v_i = \sqrt{\frac{2T}{m_i}}, \qquad v_e = \sqrt{\frac{2T}{m_e}}, \qquad T_i = T_e = T, \qquad \omega_* = k_y v_0$$

Let us consider the drift waves with phase velocity along the z-axis in the range $v_i \ll \dfrac{\omega}{k_z} \ll v_e$. In this range ion Landau damping can be neglected. For generality we allow the temperature to vary in the x-direction and the electron function to be shifted relative to the ion function by a velocity u so that there is a longitudinal electron current. In this case the dispersion relation assumes the form

$$1 - \frac{\omega_*}{\omega} - \frac{k_z^2 c_s^2}{\omega^2}\left(1 + \frac{\omega_*}{\omega}\frac{T_i}{T_e}\right) + \frac{i\sqrt{\pi}}{k_z v_e}(\omega - k_z u - \omega_* + \tfrac{1}{2}\omega_* \eta) = 0 \tag{IV. 52}$$

where $\eta = \dfrac{d\ln T_e}{d\ln n}$. For $T_i = 0$ this equation differs from (IV.37) by a small imaginary term which arises from the interaction of the wave with the resonant electrons, the longitudinal velocity of which coincides with the phase velocity of the wave. For $\omega/k_z \gg c_s$ the third term in (IV.52) can be neglected so that we have approximately

$$\omega = \omega_*, \qquad \gamma = \sqrt{\pi}\,\omega_*\left(\frac{u}{v_e} - \frac{1}{2}\frac{\omega_*}{k_z v_e}\eta\right) \tag{IV. 53}$$

For $\frac{u}{v_e} \ll 1$ and $\frac{\omega_*}{k_z} \ll v_e$ the growth rate is much less than the frequency. Moreover, in the absence of a longitudinal current the instability occurs only for $\eta = \frac{d\ln T_e}{d\ln n} < 0$. In practice, the quantity η is usually positive, and consequently long wave perturbations for $k_\perp \rho_i \ll 1$ are unstable only in the presence of a longitudinal current, although for sufficiently large values of η, to be precise for $\eta > 2$, perturbations with $k_z \sim \frac{\omega_*}{v_e}$ become unstable (20).

The instability of the drift waves for $u \neq 0$ is an extension of the ion wave instability (see (IV.2)) to the case $T_i \sim T_e$, when there is no normal ion sound. On the other hand, the instability for $u \neq 0$ can also be regarded as similar to the current-convective instability (see I.2). In effect, as in the latter case, the perturbation increases due to the drift of the particles in the electric field of an oblique wave. The transverse component of the field, which leads to the drift, arises simply as a consequence of the perturbation of the longitudinal electric field due to the interaction of the resonance electrons with the wave. The only difference between the instability considered here and the current-convective instability is that in this case the effects of collisions are produced by Landau damping.

(c) *Drift Instability for* $k_\perp \rho_i \gtrsim 1$

It is evident from (IV.52) that for $\eta = 0$ and $u = 0$ the growth rate vanishes only because of what would appear to be an accidental circumstance, namely that the frequency of the oscillations exactly coincides with ω_*. Any effect which shifts the oscillation frequency from the value ω_* leads to growth or damping of the drift waves. In a plasma with cold ions ($T_i = 0$) it may be necessary not to use the drift approximation for the ions but to include the inertia term in the equation of motion, i.e. the second term in (IV.34). When the corresponding change in the expression for the ion density is made, it can be shown that eqn. (IV.52) takes the following form

$$\frac{k_\perp^2 T_e}{m_i \Omega_i^2} + 1 - \frac{\omega_*}{\omega} - \frac{k_z^2 c_s^2}{\omega^2} + \frac{i\sqrt{\pi}}{k_z v_e}(\omega - k_z u - \omega_* + \tfrac{1}{2}\omega_* \eta) = 0 \quad \text{(IV. 54)}$$

The transverse inertia term becomes important for $k_\perp r_H \sim 1$, where r_H is the Larmor radius of the ions at the electron temperature. There is then an instability even in a currentless plasma for $\eta = 0$.

When the ions are hot a similar effect occurs for $k_\perp \rho_i \sim 1$. For $T_e = T_i = T$ and $\eta = 0$ the dispersion equation for longitudinal oscillations ($\omega \ll k_z c_A$) can be obtained by solving the kinetic equations for both the electrons and the ions and has the form

$$e^{-s} I_0(s) \frac{\omega + \omega_*}{k_z v_i} Y\left(\frac{\omega}{k_z v_i}\right) + \frac{\omega - \omega_*}{k_z v_e} Y\left(\frac{\omega}{k_z v_e}\right) = 2 \quad \text{(IV. 55)}$$

where $v_i = \sqrt{\dfrac{2T}{m_i}}$, $v_e = \sqrt{\dfrac{2T}{m_e}}$, $s = k_\perp^2 \rho_i^2 = \dfrac{k_\perp^2 T}{m_i \Omega_i^2}$, I_0 is the Bessel function of imaginary argument, and Y is the function defined by eqn. (IV.49). Equation (IV.55) differs from (IV.51) only in that in the first term an additional factor $e^{-s}I_0(s)$, appears, which represents the decrease of the ion drift velocity due to the averaging of the wave field over the Larmor orbit (see Section (d)).

This equation was investigated in ref. (83); the results were that only the accelerated sound waves, which propagate in the direction of the electron drift (Curve 1 on Fig. 17), are unstable. For $v_i \ll \dfrac{\omega}{k_z} \ll v_e$ in (IV.55) we may put $Y\left(\dfrac{\omega}{k_z v_i}\right) \cong \dfrac{k_z v_i}{\omega}$, $Y\left(\dfrac{\omega}{k_z v_e}\right) \cong -i\sqrt{\pi}$ and we then obtain

$$\omega = \omega_* \frac{\beta_s}{2-\beta_s}, \qquad \gamma = 2\sqrt{\pi}\frac{\omega_*^2}{k_z v_e}\cdot\frac{\beta_s(1-\beta_s)}{(2-\beta_s)^2} \qquad \text{(IV. 56}$$

where $\beta_s = e^{-s}I_0(s)$, and for $k_z v_e \gg \omega_*$ the growth rate is small compared to the frequency. However, if we decrease k_z until the phase velocity of the wave parallel to the z axis approximates to v_e, the growth rate γ becomes of

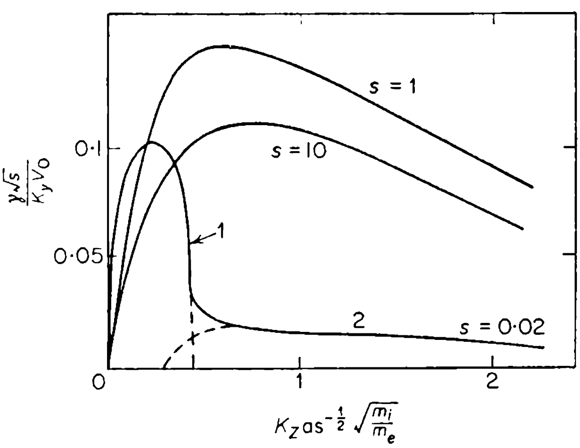

FIG. 19. Dependence of growth rate γ on longitudinal wave number k_z for the drift instability

the order ω. The results of the numerical calculation of the growth rate as a function of k_z near the maximum are shown in Fig. 19 for three values of s and for $k_x = 0$.

For small s the unstable oscillations can be divided into two branches. The frequency of one of the branches is given by the relation (IV.56) and the corresponding growth rate decreases as s^2 for $s \to 0$. The frequency of the second branch can be obtained by using the asymptotic expansion

$Y(z) \approx \frac{1}{z} + \frac{1}{2z^3}$ for $z \gg 1$. Equation (IV.55) then becomes the cubic

$$\frac{\omega - \omega_*}{\omega + \omega_*} = \frac{2\omega^2 s}{k_z^2 v_e^2} = \frac{\omega^2}{\omega_z^2} \tag{IV. 57}$$

where $\omega_z^2 = \Omega_i \Omega_e \frac{k_z^2}{k_\perp^2}$. For $\frac{\omega_*}{\omega_z} > 0{\cdot}4$ complex roots appear in this equation and the corresponding growth rate is of the order of ω_*. On Fig. 19 section 1 corresponds to this instability.

Figure 20 shows the boundary of instability for $k_x = 0$ and $m_i/m_e = 1840$.

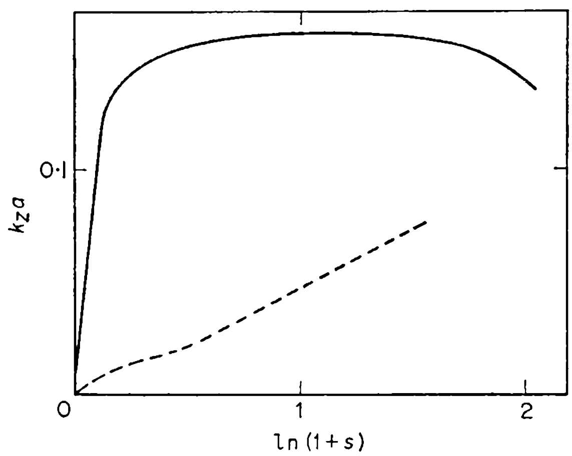

FIG. 20. Region of drift instability in a low pressure plasma (unstable below the solid curve). For dashed curve see text

For small s it is defined by the condition $\omega \leqslant \omega_*$ and is given approximately by $ak_z = s$, whilst for $s \gtrsim 1$ the limiting value of $k_z a$ is determined by the ion damping and is almost independent of s. The dashed curve on this figure is the locus of maximum growth rate as a function of $k_z a$.

The above relations refer to the case $\omega < k_z c_A$, i.e. they are correct for the whole region of k_z only if $v_e \ll c_A$, and consequently $\beta \ll m_e/m_i$. In the opposite case $\beta \gtrsim m_e/m_i$, we must consider the curvature of the lines of force when k_z is small. Mikhailovskiĭ and Rudakov (84) have shown that in this case the instability develops only on the accelerated sound branch (Curve 1 on Fig. 18). For $\eta = 0$ and $s \gtrsim 1$ the frequency of these oscillations is given approximately by

$$\omega \approx \frac{\omega_*}{2} \frac{\beta_s \zeta^2}{\zeta^2 + k_y^2/k_\perp^2} \tag{IV. 58}$$

and the growth rate γ by

$$\gamma \approx \frac{\sqrt{\pi}\,\omega^2}{|k_z| v_e \beta_s} \tag{IV. 59}$$

where $\zeta = 2k_z a/\sqrt{\beta}$. The growth rate γ as a function of k_z reaches a maximum $\sim \frac{\omega}{\beta_s}\sqrt{\frac{m_e}{m_i \beta}}$ for $\zeta \approx 1$. As $k_\perp \rho_i$ increases, the frequency of the oscillations remains approximately constant, the growth rate increases and for $s \sim \frac{m_i}{m_e}\beta$ it becomes of the order of the frequency.

These results refer to the case of constant temperature. However, when we include a temperature gradient the instability still exists at practically any given ratio between the density and temperature gradients (85, 86).

As β increases, the ion thermal velocity approaches the Alfvén speed and the ion Landau damping becomes more and more important. At sufficiently large β values, as has been shown in ref. (93), this effect completely stabilises the drift wave. Figure 21 taken from ref. (93) shows the critical β

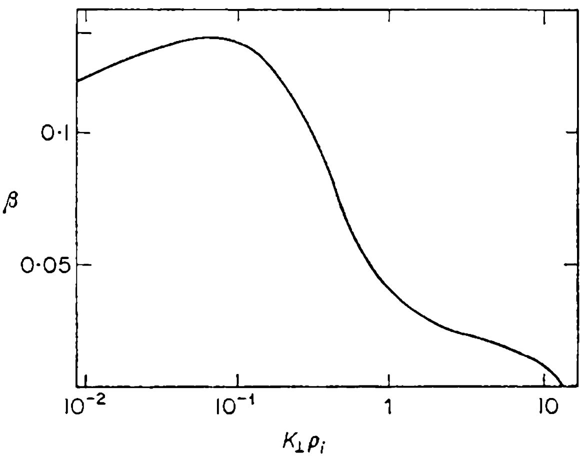

FIG. 21. Region of drift instability as a function of β (unstable below the curve)

at which the oscillations become stable as a function of $k_\perp \rho_i$ (stability occurs above the curve). For $\beta > 0{\cdot}13$ the drift instability is completely suppressed by the ion damping.

In addition the drift instability must be absent in systems which are not very long. For, as is apparent from Fig. 20, the drift instability develops only for perturbations with small k_z; more precisely, $k_z a$ must be of the order of 10^{-1}. In devices of limited length the wave number k_z cannot be small and the plasma must be stable.

A further effect contributing to the stabilisation of the drift instability is the shear of the lines of force, refs. (88, 89). We shall characterize the magnitude of the shear by the parameter $\theta = a/L$, where a is the transverse dimension of a plasma column and L is a length along the column such that the rotational transform angle referred to this length (94) varies across the column by a value of order unity. Broadly speaking θ is the angle between

two lines of force separated from each other by a distance a. The projection of the wave vector on to the line of force varies across the column by about $k\theta$. If we assume $k_\perp \sim \rho_i^{-1}$, then even for the very small value $\theta \sim \rho_i/a$ we have $k\theta \sim a^{-1}$, so that at some point in the column we must have a region where $k_z \sim a^{-1}$, i.e. a region of strong absorption at the ions. Thus, for $L \sim a^2/\rho_i$ we expect stabilisation due to shear. For some perturbations the value of θ necessary for stability may be slightly larger than ρ_i/a because of the reflection of the wave at " potential barriers ", i.e. turning points where for the frequency ω, k_x becomes imaginary (see refs. 88, 89). But even in this case the values of θ required for stability are very small.

(d) *The Stabilisation of the Convective (Flute) Instability*

As we showed above, the inhomogeneity of the plasma leads to a considerable change in the dispersion relation $\omega(k)$ in the region of small frequencies of the order of ω^*. In other words, for oscillations with phase velocity of the order of the drift velocity the properties of the plasma are appreciably different from those of a conducting fluid described by the magnetohydrodynamic equations. Rosenbluth *et al.* (21) have shown that the drift effects may, in particular, considerably influence the magnetohydrodynamic flute instability of a plasma and under certain conditions complete stabilisation of this instability appears possible.

Qualitatively the possibility of this stabilisation may be seen directly from Fig. 18. From the magneto-hydrodynamic point of view, the frequency of Alfvén oscillations is given by $\omega = k_z c_A$ and for $k_z = 0$ it becomes zero. Perturbations with $k_z = 0$ remain constant along the lines of force of the magnetic field and it is these perturbations which are referred to as flute perturbations. The fact that the frequency of the oscillations vanishes means that for these perturbations the plasma shows no " elasticity " whatever, and they can therefore be destabilised by any curvature of the lines of force or any gravitational force acting in the direction of decreasing density, no matter how small. When we consider the drift effects, however, we can see from Fig. 18 that the relationship between the oscillation frequency and k_z becomes more complex and the transverse motion cannot be considered completely inelastic.

Let us consider this effect in some detail. Suppose that a cold plasma ($T_i = T_e = 0$) is situated in a strong magnetic field acting parallel to the z axis, and subject to the effect of a gravitational force with acceleration $\mathbf{g}$. We shall assume that $\mathbf{g}$ is parallel to the x axis and in the direction of decreasing density. In the equilibrium state, in a co-ordinate system where the mean electric field vanishes, the electrons are at rest and the ions drift parallel to the y axis with a velocity $v_{i0} = g/\Omega_i$. In the perturbed plasma the velocity of the electrons is determined by the electric drift $\frac{c}{H}[\mathbf{h}\nabla\varphi]$, and the perturbation of the ion velocity v_i can be determined using the equation of motion which for perturbations of the form $\exp(-i\omega t + i\mathbf{kr})$ takes the following form

$$-i\omega' m_i \mathbf{v}_i = -ie\mathbf{k}\varphi + \frac{e}{c}[\mathbf{v}_i \mathbf{H}] \tag{IV. 60}$$

where $\omega' = \omega + k_y g/\Omega_i$. For $\omega' \ll \Omega_i$ we obtain approximately

$$\mathbf{v}_i = \frac{ic}{H}[\mathbf{hk}]\varphi + \frac{c}{H}\frac{\omega'}{\Omega_i}\mathbf{k}_\perp \varphi \tag{IV. 61}$$

and substituting this expression into the continuity equation for the ions, we obtain

$$-i\omega' n_i - i\frac{ck_y}{H}\frac{dn}{dx}\varphi + in\frac{c\omega' k_\perp^2}{H\Omega_i}\varphi = 0 \tag{IV. 62}$$

and the perturbation of the ion density is given by

$$n_i/n = \frac{ck_y \kappa}{\omega' H}\varphi - \frac{k_\perp^2}{\Omega_i^2}\frac{e\varphi}{m_i} \tag{IV. 63}$$

For the electron density perturbation we obtain instead of (IV.63) the much simpler expression

$$n_e/n = \frac{ck_y \kappa}{\omega H}\varphi \tag{IV. 64}$$

because we can neglect the electron inertia term.

Substituting the values for n_i and n_e so obtained into the equation $k^2\varphi = 4\pi e\,(n_i - n_e)$ and assuming for simplicity $k_x = 0$, we obtain

$$\frac{\Omega_i \kappa}{k}\left(\frac{1}{\omega + k_y g/\Omega_i} - \frac{1}{\omega}\right) - 1 = \frac{c_A^2}{c^2} \tag{IV. 65}$$

The terms in brackets are due to the difference between the drift velocities of the electrons and the ions arising from the gravitational field, the unit represents the inertia term in (IV.62) and the right hand side represents the effect of possible departures from quasi-neutrality. In a dense plasma, where $c_A \ll c$ this right hand side can be neglected.

There is a large factor of approximately Ω_i in front of the brackets in (IV.65) and therefore the oscillation frequency is considerably larger than $k_y g/\Omega_i$. Expanding $\left(\omega + \frac{k_y g}{\Omega_i}\right)^{-1}$ to first order in this quantity, we obtain from (IV.65) the much simpler dispersion equation $\omega^2 + g\kappa = 0$ and thus $\omega = \pm i\sqrt{g\kappa}$ so that the plasma is convectively unstable.

According to (IV.65), the instability originates from the small difference between the drift velocities of the electrons and the ions. Naturally any additional factors which may influence this velocity difference must also considerably influence the stability of the plasma. One of these factors is the effect of the finite Larmor radius of the ions, and including this, using (IV.50) for small s, we obtain, instead of (IV.63)

$$\frac{n_i}{n} = \left(\frac{\omega_*}{\omega'}(1-s) - s\right)\frac{e\varphi}{T} \tag{IV. 66}$$

This shows that the effect of the finite Larmor radius of the ions can be included in the dispersion eqn. (IV.65) by simply introducing the factor $1-s$ in the first term in the round brackets. For $c_A \ll c$ the equation can then be represented in the form

$$\omega^2+\omega_*\omega+g\kappa = 0 \tag{IV. 67}$$

whence we obtain

$$\omega = -\frac{\omega_*}{2} \pm \sqrt{\frac{\omega_*^2}{4} - \kappa g} \tag{IV. 68}$$

This shows that for $g\kappa < \frac{\omega_*^2}{4}$ the plasma is convectively stable. If we introduce the effective radius of curvature instead of g according to the equivalence $g = \frac{T}{m_i R}$, the stability condition takes the form

$$k^2\rho_i^2 > \frac{4a}{R} \tag{IV. 69}$$

where $\rho_i^2 = \frac{T}{m_i \Omega_i^2}$ and $a = \kappa^{-1}$. Thus for $a \ll R$ which is often the case in practice, even perturbations the transverse length of which considerably exceeds ρ_i will be stable.

For $s \gtrsim 1$, Mikhailovskiĭ (95) has shown that there is also a slight change in the first term in the dispersion equation for the flute perturbations, and the dispersion equation valid for all s has the following form

$$\frac{\Omega_i \kappa}{k}\left(\frac{e^{-s}I_0(s)}{\omega+k_y g/\Omega_i} - \frac{1}{\omega}\right) - \frac{1-e^{-s}I_0(s)}{s} = \frac{c_A^2}{c^2} \tag{IV. 70}$$

The result of the investigation of this equation carried out in (90) is shown on Fig. 22. As this figure shows, in a dense plasma ($c_A \ll c$) all

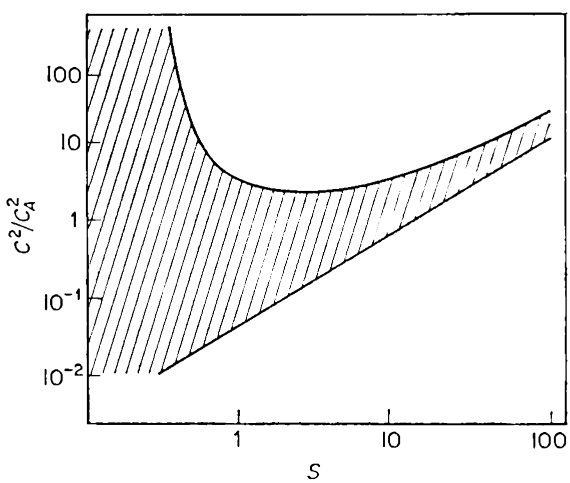

FIG. 22. Region of flute instability (hatched)

perturbations with sufficiently short wavelengths are stabilised, and the stability condition is that given by the relation (IV.69). As the density decreases the plasma becomes unstable, but for a very small density the stability is again restored, since ultimately a transition will take place to separate non-interacting particles which can no longer be described as a plasma. In the intermediate region there is a gap where the plasma is unstable. This gap extends into the high density region, but only for extremely short wavelength perturbations, which cannot make any appreciable contribution to the transverse diffusion.

We considered earlier only short wave perturbations for which the quasi-classical approximation is valid. Qualitatively the conclusion that the flute instability may be stabilised also refers to long wave perturbations except the so-called $m = 1$ mode, corresponding to the transverse displacement of the plasma as a whole. With this type of displacement the perturbed electric field in the filament is almost constant over the cross-section so that the drift velocities of the electrons and ions coincide closely and the stabilisation due to the finite Larmor radius is absent.

An instability of the $m = 1$ mode is more conveniently described by the momentum equation. For $c_A \ll c$ the momentum of the electromagnetic field and the electrostatic forces can be neglected, and the equations for the momentum for the displaced plasma filament can be written in the following form:

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} \tag{IV. 71}$$

where $\mathbf{P} = m_i N \dfrac{d\mathbf{r}_c}{dt}$ is the momentum of the filament, N the total number of ions per unit length, $\mathbf{r}_c$ the radius vector of the mass centre and $\mathbf{F}$ the gravitational force. Suppose that $\mathbf{g}$ increases linearly with distance from the symmetry axis: $\mathbf{g} = b\mathbf{r}$. Then the force $\mathbf{F} = m_i N b \mathbf{r}_c$ and we obtain from (IV.71) $\ddot{\mathbf{r}}_c = b\mathbf{r}$ or $\omega^2 = -b$, and consequently the plasma filament is unstable relative to this displacement. In an actual case of plasma in a trap with inhomogeneous magnetic field, the force $\mathbf{F}$ is determined by the integral over the volume of the quantity $\dfrac{(T_i+T_e)n}{R}$, where n is the electron density and R the mean radius of curvature of the lines of force. When the mean curvature of the lines of force is proportional to the distance from the symmetry axis, i.e. $\dfrac{1}{R} = \dfrac{r}{aR_0}$, where a and R_0 are constants, the force $\mathbf{F}$ can be represented in the form $\mathbf{F} = \dfrac{P_0}{aR_0}\mathbf{r}_\mathrm{T}$, where $P_0 = \int p d\mathbf{r} \approx$ const, and $\mathbf{r}_\mathrm{T} = P_0^{-1}\int \mathbf{r} p \, d\mathbf{r}$ is the radius vector of the " pressure centre " of the plasma. If the temperature of the plasma is constant over the cross-section, $\mathbf{r}_\mathrm{T}$ is equal to $\mathbf{r}_c$. We then again have $\ddot{\mathbf{r}}_c = b\mathbf{r}_c$, so that the plasma is unstable.

(e) *Ion-sound Instability of an Inhomogeneous Plasma*

We have considered so far only oscillations with frequencies much less than both the electron and the ion cyclotron frequencies, so that both the electrons and ions were " magnetised ". Let us now consider the other limiting case where the magnetic field is so weak that its effect on the ions can be neglected (90). In other words, we shall consider oscillations with a frequency considerably larger than Ω_i, but we shall still assume that the frequency is less than Ω_e, so that for the electrons we may use the drift approximation (we assume $k\rho_e \ll 1$). For simplicity the ion temperature is assumed to be zero. We shall limit the discussion to the case of a low pressure plasma in a homogeneous magnetic field and to " potential " oscillations for which $\mathbf{E} = -\nabla\varphi$. With these assumptions the equation of motion for the ions, which we shall suppose at rest in the equilibrium state, takes the form

$$\omega m_i \mathbf{v}_i = e\mathbf{k}\varphi \tag{IV. 72}$$

from which, together with the continuity equation

$$\omega n_i - \mathbf{k}\mathbf{v}_i n = 0 \tag{IV. 73}$$

we obtain the perturbation of the ion density

$$n_i/n = \frac{k^2}{\omega^2}\frac{e}{m_i}\varphi \tag{IV. 74}$$

The density of the electrons is determined by relation (IV.50). Comparing n_i and n_e we obtain the dispersion relation for these oscillations

$$\frac{k^2 T_e}{m_i\omega^2} + \frac{\omega-\omega_*}{k_z v_e} Y\left(\frac{\omega}{k_z v_e}\right) = 1 \tag{IV. 75}$$

For oscillations with phase velocity $\omega/k_z \ll v_e$ this assumes the simpler form

$$\frac{k^2 c_s^2}{\omega^2} - i\sqrt{\pi}\frac{\omega-\omega_*}{k_z v_e} = 1 \tag{IV. 76}$$

From this we obtain the frequency of the oscillations $\omega = kc_s$ and the growth rate

$$\gamma = \frac{\sqrt{\pi}k}{k_z}\sqrt{\frac{m_e}{m_i}}(\omega_*-\omega) \tag{IV. 77}$$

From this relation we see that for $\omega < \omega_*$ the plasma becomes unstable. We have already established that the same condition applies to the growth of drift waves. In the case considered here, the condition means that the drift velocity of the electrons must be higher than the velocity of sound. It can also be put in the form

$$r_H \kappa > 1 \tag{IV. 78}$$

where $r_H = \sqrt{\dfrac{T_e}{m_i\Omega_i}}$ is the Larmor radius of the ions calculated from the electron temperature.

According to (IV.76), the growth rate increases as k_z decreases. This increase continues as the longitudinal phase velocity approaches the thermal velocity of the electrons, i.e. almost to $k_z/k \sim \sqrt{\dfrac{m_e}{m_i}}$, when the growth rate attains its maximum value $\gamma \sim \omega \sim kc_s$. For a further decrease of k_z both the frequency of the oscillations and the growth rate decrease.

(f) *Cyclotron Instability of an Inhomogeneous Plasma*

When the magnetic field increases condition (IV.78) is infringed, and the ion-sound instability goes over into the drift instability considered earlier, which develops for oscillations with a frequency considerably smaller than Ω_i. Mikhailovskiĭ and Timofeev (96), (97) have shown also that oscillations with frequencies near multiples of the cyclotron frequency, $\omega = n\Omega_i$, may also be unstable in an inhomogeneous plasma. From this point of view, the drift instability can be regarded as a cyclotron instability with $n = 0$, and the ion sound instability as an instability at very high harmonics of the cyclotron frequency.

For oscillations with a frequency close to $n\Omega_i$, when the contribution of the other harmonics to the density perturbation can be neglected, the dispersion equation for $T_i = T_e$ takes the form

$$\frac{\omega+\omega_*}{k_z v_i} Y\left(\frac{\omega - n\Omega_i}{k_z v_i}\right) e^{-s} I_n(s) + \frac{\omega-\omega_*}{k_z v_e} Y\left(\frac{\omega}{k_z v_e}\right) = 2 \qquad \text{(IV . 9)}$$

where I_n is the Bessel function of imaginary argument of order n, and the other symbols are the same as those introduced earlier (see (IV.55)). For $k\rho_e \gtrsim 1$, we must introduce the additional factor $e^{-k^2\rho_e^2} I_0(k^2\rho_e^2)$ in the second term on the left hand side of eqn. (IV.79). In addition, for very large values of the wave number k the quasi-neutrality condition may be infringed, and it is necessary to add the term k^2D^2 to the right hand side of eqn. (IV.79).

Equation (IV.79) can be considerably simplified for small k_z, where $Y\left(\dfrac{\omega - n\Omega_i}{k_z v_i}\right) \approx \dfrac{k_z v_i}{\omega - n\Omega_i}$. The condition for the growth of such oscillations can easily be shown to have the form obtained previously, $\omega < \omega_*$, i.e.

$$\kappa k \rho_i^2 > n \qquad \text{(IV. 80)}$$

when waves propagating in both the electron drift and the ion drift directions are unstable.

It is interesting to note that cyclotron oscillations can also be built up for purely transverse propagation ($k_z = 0$) if the following condition is fulfilled

$$\kappa\rho_i \geqslant 2n\left(\frac{m_e}{m_i} + \frac{c_A^2}{c^2}\right)^{\frac{1}{4}} \qquad \text{(IV. 81)}$$

where n is the number of the harmonic considered. In the case of a rarefied

plasma, the quantity $\frac{m_e}{m_i}$ underneath the root sign in (IV.81) can be neglected and for $n = 1$ this condition takes the following form

$$\kappa\rho_i \geqslant 2c_A/c \tag{IV. 82}$$

The growth rate of the cyclotron instability may reach values of the order $\left(\frac{m_e}{m_i}\right)^{\frac{1}{4}}\Omega_i$. If the condition (IV.82) is not fulfilled, an exponentially small "residual" instability may develop (98).

(g) *Drift Dissipative Instability*

In (b) and (c) we considered the growth of drift waves in a collisionless plasma. It has also been found that the drift instability may develop in a dense plasma in which an important part is played by collisions between the particles (23), (24), (90), (92), and the collisions are in fact responsible for the instability. To distinguish between this instability and the collisionless instability, we shall call the former the drift-dissipative instability.

We assume that the electron mean free path λ_e is considerably smaller than the longitudinal wavelength, so that the diffusion approximation can be used for the longitudinal motion. For simplicity we shall limit the discussion to the case of a plane layer of plasma, although in fact we have in mind a cylindrical plasma column in a tube of radius $a \sim \kappa^{-1}$. The transverse length of the perturbations will be assumed to be considerably smaller than a.

Let us start with the case of a weak magnetic field, assuming that the ions are cold and the Larmor radius calculated from the electron temperature $r_H \gg a$. We assume furthermore that in equilibrium there is no mean electric field and the ions are at rest. These conditions may occur in practice in a positive column at a very low neutral gas pressure. If the mean free path length of the ions λ_i is of the same order as λ_e, and is larger than a, then the ion collisions can be neglected and for oscillations with frequency $\omega \gg \Omega_i$ we have from (IV.74) $n_i/n = \frac{k^2 e\varphi}{\omega^2 m_i}$. The perturbation of the electron density is obtained from the continuity equation

$$\omega n_e + \frac{ck_y}{H}\frac{dn}{dx}\varphi - k_z v_{ze} n = 0 \tag{IV. 83}$$

where for $\mathbf{v}_\perp$ we substitute the electric drift velocity, assuming that $\Omega_e\tau_e \gg 1$, so that the transverse diffusion and mobility of the electrons can be neglected. The longitudinal velocity of the electrons in diffusion conditions is determined by the relation

$$v_{ze} = b_e\frac{\partial\varphi}{\partial z} - D_e n^{-1}\frac{\partial n_e}{\partial z} = \frac{e\tau_e}{m_e}\left(\frac{\partial\varphi}{\partial z} - \frac{T_e}{en}\frac{\partial n_e}{\partial z}\right) \tag{IV. 84}$$

where b_e is the mobility, D_e the diffusion coefficient and τ_e the mean collision time between electrons and neutrals or ions (for collisions with ions it is necessary to substitute the relative velocity for v_{ze} in (IV.84), but since the

longitudinal motion of the ions in these oscillations can be neglected, this effect is not important).

Substituting the expression for v_{ze} into the continuity equation for the electrons, (IV.83), we obtain

$$\frac{n_e}{n} = \frac{k_z^2 - i\dfrac{k_y\kappa}{\Omega_e\tau_e}}{D_e k_z^2 - i\omega} \cdot \frac{e\tau_e}{m_e}\varphi \tag{IV. 85}$$

For $\omega \ll D_e k_z^2$ this expression can be simplified to

$$\frac{n_e}{n} = \left(1 - \frac{ik_y\kappa}{\Omega_e\tau_e k_z^2}\right)\frac{e\varphi}{T_e} \tag{IV. 86}$$

Comparing n_i and n_e we arrive at the dispersion equation

$$\omega^2 = k^2 c_s^2\left(1 - \frac{ik_y\kappa}{\Omega_e\tau_e k_z^2}\right)^{-1} \tag{IV. 87}$$

The frequency of the oscillations is complex and one of the roots has a positive imaginary part, so that the plasma is unstable. Since $\omega \sim kc_s$ this instability may also be classified as ion-sound instability. It is evident from (IV.87) that for $\Omega_e\tau_e \gg 1$ perturbations will be built up in the first place which are strongly elongated parallel to the magnetic field, i.e. $k_z \ll k_y$.

The growth rate γ attains a maximum $\gamma \sim \omega \sim c_s k$ for $k_z^2 = k_y\kappa/\Omega_e\tau_e$. Then $k_z^2 D_e\omega^{-1} = \kappa r_H \gg 1$, so that the assumption $\omega \ll k_z^2 D_e$ is fulfilled throughout except for the region of very small k_z which is not of great interest.

For $r_H\kappa \lesssim 1$ the longitudinal diffusion of the electrons must be considered in (IV.85). It can be shown (90) that in this case the plasma is stable to acoustic oscillations with frequency $\omega \gg \Omega_i$.

A plasma with these parameters can be set up, for instance, in the positive column of a glow discharge. In stationary conditions such a plasma will diffuse to the wall, and if the diffusion speed is of the order of c_s, these oscillations cannot develop before reaching the wall, since their group velocity is also of the order c_s. Since the transverse diffusion coefficient $D_\perp$ is of the order of $\lambda_e v_e(\Omega_e\tau_e)^{-2}$, the growth condition $D_\perp\kappa \ll c_s$ can be represented approximately in the following form

$$\lambda_e > \frac{r_H^2}{a}\sqrt{\frac{m_e}{m_i}} \tag{IV. 88}$$

where λ_e is the electron mean free path, $r_H^2 = \dfrac{T_e}{m_i\Omega_i^2}$, $a = \kappa^{-1}$. Thus this instability occurs, in the absence of a longitudinal current, only in a sufficiently rarefied plasma where the electron mean free path is not too small.

For $\lambda_i \lesssim a$ we must also consider collisions between ions and neutral gas atoms. This effect (85) leads to the instability condition $\Omega_i\tau_i > \sqrt{\dfrac{b_i}{b_e}}$,

where b_e is the electron and b_i the ion mobility. When the magnetic field decreases condition (IV.88) is replaced by the inequality $\rho_e < a$, where ρ_e is the mean Larmor radius of the electrons, since for $\rho_e > a$ the effect of the magnetic field on the electrons can be neglected, and there will be no instability.

Let us now consider waves with frequency $\omega \ll \Omega_i$. The perturbation of the electron density is again given by (IV.85), and including the effect of friction between ions and neutral gas, the perturbation of the ion density is given by

$$\frac{n_i}{n} = \frac{ck_y\kappa}{\omega H}\varphi - \frac{k_\perp^2}{\Omega_i^2}\left(1 + \frac{i}{\omega\tau_i}\right)\frac{e\varphi}{m_i} \tag{IV. 89}$$

for $\Omega_i\tau_i \gg 1$, where τ_i is the mean collision time of ions with neutral gas atoms (see (IV.63)). Comparing n_e and n_i we obtain the dispersion equation which can be conveniently written

$$\omega^2 + i\omega\left(\omega_s + D_e k_z^2 + \frac{1}{\tau_i}\right) - i\omega_s\omega_* - \frac{D_e k_z^2}{\tau_i} = 0 \tag{IV. 90}$$

where

$$\omega_* = -k_y\frac{cT_e}{eHn}\frac{dn}{dx}, \qquad \omega_s = \frac{k_z^2}{k_\perp^2}\Omega_e\tau_e\Omega_i$$

It is readily seen that $D_e k_z^2/\omega_s = \dfrac{k_\perp^2 T_e}{m_i\Omega_i^2} = k_\perp^2 r_H^2$. Therefore in a strong magnetic field, when $\kappa r_H \ll 1$, for perturbations with $k_\perp r_H < 1$ we may neglect $D_e k_z^2$ compared with ω_s. If in addition the collision frequency $\nu_i = \tau_i^{-1}$ is sufficiently small, then (IV.90) can be simplified to

$$\omega^2 + i\omega\omega_s - i\omega_s\omega_* = 0 \tag{IV.91}$$

This equation is also valid for a fully-ionised plasma (24), if by τ_e^{-1} we understand the mean collision frequency of electrons with ions. It follows from eqn. (IV.91) that for $\omega_s \gg \omega_*$

$$\omega_1 = \omega_* + i\frac{\omega_*^2}{\omega_s}, \qquad \omega_2 = -i\omega_s \tag{IV. 92}$$

and for $\omega_s \ll \omega_*$

$$\omega_{1,2} = \pm\sqrt{i\omega_s\omega_*} \tag{IV. 93}$$

In either of these limiting cases one of the roots has a positive imaginary part, giving an instability. For $\omega_s \sim \omega_*$ the growth rate attains its maximum value $\gamma \sim \omega \sim \omega_*$.

In a weakly ionised plasma the collisions between ions and neutrals give rise to an additional damping, and for $\kappa c_s\tau_i < 1$ all perturbations are damped (90).

In the other limiting case $\kappa r_H \gg 1$ drift waves with frequency $\omega < \Omega_i$ may grow as well as ion-sound waves, but their growth rate, which is by definition smaller than Ω_i, is therefore smaller than the growth rate of the

ion-sound oscillations. Drift waves can grow only for $\Omega_i\tau_i > 1$, when the drift velocity $\frac{c[\mathbf{h}\nabla\varphi]}{H}$ is larger than $b_i\nabla\varphi$.

The entire region of the drift-dissipative instability in a weakly ionised plasma is represented schematically in Fig. 23. The abscissa is $\eta = r_H^{-1}\kappa^{-1}$

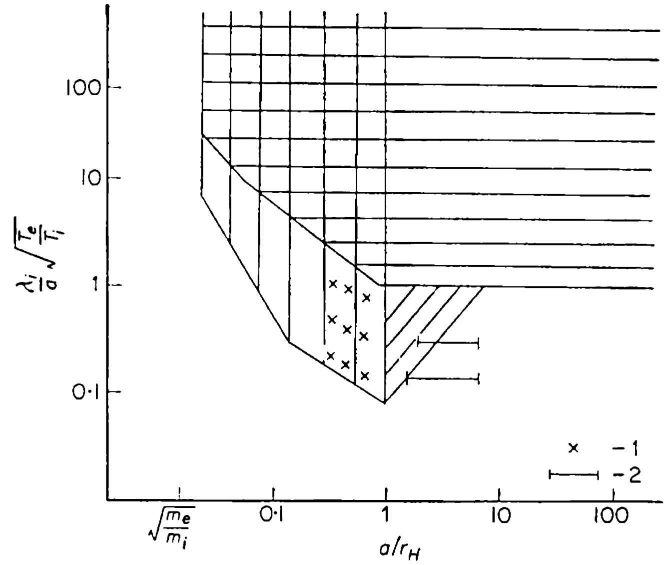

FIG. 23. Region of ion acoustic (vertical hatching) and drift (horizontal hatching) instabilities in an inhomogeneous weakly-ionised plasma. For diagonal hatching see text. (1) results of Geller (131), (2) results of Golant (111)

which is proportional to the magnetic field, and the ordinate the quantity $\xi = \sqrt{\frac{T_e}{T_i}}\tau_i\kappa$ which is proportional to $1/ap$, where p is the neutral gas pressure and a the tube radius. For η less than unity but greater than $\sim\sqrt{\frac{m_e}{m_i}}$, where the Larmor radius of the electrons is of the order a, an ion-sound instability will develop. The corresponding region is marked in Fig. 23 by the vertical hatching. The region of the drift instability proper $(\omega < \Omega_i)$ is marked by horizontal hatching. For $\eta > 1$ it has a lower limit $\xi = 1$, but for $\eta < 1$ the instability develops only for $\Omega_i\tau_i > 1$.

Thus, for sufficiently low neutral gas pressure, a weakly ionised plasma in a homogeneous magnetic field is unstable even in the absence of a longitudinal current. In the presence of a longitudinal current, the instability region expands towards smaller ξ values, since even for $\lambda_i < a$ the current-convective (spiral) instability previously discussed may develop in a plasma with a longitudinal current (see I.2.b). Moreover, the longitudinal current may also have an effect on the drift-dissipative instability. In the presence of a

longitudinal current the expression for the perturbation of the electron density changes to

$$\frac{n_e}{n} = \frac{k_z^2 + ik_y\kappa(\Omega_e\tau_e)^{-1}}{D_e k_z^2 - i\omega + ik_z u} \cdot \frac{e\tau_e}{m_e}\varphi \tag{IV. 94}$$

where u is the longitudinal (current) velocity of the electrons.

Accordingly, in expression (IV.87) for the square of the frequency of the ion-sound oscillations, we obtain an extra factor $1 + iu/D_e k_z^2$, and in this case the maximum value of the growth rate again occurs for $k_z^2 \sim k_y\kappa/\Omega_e\tau_e$. For a sufficiently large $\Omega_e\tau_e$ the maximum growth rate may increase by a factor $\left(\frac{u^2\Omega_e\tau_e}{D_e k_y \kappa}\right)^{\frac{1}{4}}$ compared with the maximum value $\gamma \sim kc_s$ obtained previously.

The longitudinal current also strengthens the intrinsic drift instability. Using eqn. (IV.94) the dispersion equation becomes more complex than eqn. (IV.91), taking the form

$$\omega^2 + i\omega(\omega_s + ik_z u) - i\omega_* \left(\omega_s + i\frac{k_z u}{k_\perp^2 r_H^2}\right) = 0 \tag{IV. 95}$$

where, just as before $\omega_s = \frac{k_z^2}{k_\perp^2}\Omega_e\tau_e\Omega_i$, $\omega_* = c_s k_y \kappa r_H$.

As we established earlier, in the absence of a current the increment attains its maximum for $\omega_s \sim \omega_*$, i.e. for $k_z^2 \sim k_\perp^3 \kappa r_H^2 (\Omega_e\tau_e)^{-1}$. The effect of the current becomes appreciable if at the maximum the value of $k_z u/k_\perp^2 r_H^2$ is comparable with ω_s i.e. for $u > u_c = c_s r_H^2 k_\perp^2 \sqrt{\frac{\kappa}{k_\perp}\Omega_e\tau_e}$. We assume that u exceeds this value. Then the point of maximum growth rate shifts towards large k_z, since the value of uk_z increases with k_z, and the actual maximum value slightly increases compared with the currentless plasma.

For sufficiently large k_z values, when $\omega_s > k_z u$ we have approximately

$$\omega \cong \omega_*\left(1 + \frac{ik_z u}{\omega_s k_\perp^2 r_\kappa^2}\right) = \omega_*\left(1 + i\frac{u}{v_e k_z \lambda_e}\right), \tag{IV. 96}$$

where λ_e is the electron mean free path. We must assume $k_z\lambda_e < 1$ because otherwise the diffusion approximation could not be used for the longitudinal motion of the electrons. Equation (IV.96) shows that as k_z decreases the growth rate increases. For small $r_H k_\perp$ this increase continues until ω_s^2 attains values of the order $\omega_* k_z u/k_\perp^2 r_H^2$, and then the growth rate begins to decrease with k_z. The maximum value of the growth rate, of the order of

$$\gamma_1 = u\kappa\left(\frac{c_s^2}{u_0 v_e \lambda_e \kappa}\right)^{\frac{1}{3}} \tag{IV. 97}$$

is reached for $k_z \sim \frac{k_\perp \rho_i}{\lambda_e}\left(\frac{u\kappa\lambda_e c_s}{v_e^2}\right)^{\frac{1}{3}}$.

For not very small values of $k_\perp r_H$ it may be found that the value of $k_z u$ becomes larger than ω_s before we reach this value of k_z. In this case we can neglect ω_s compared with $k_z u$ in the dispersion equation (IV.95), and then the maximum value is

$$\gamma_2 = \omega_* / k_\perp^2 r_H^2 \tag{IV. 98}$$

for $k_z u \sim \omega_*/k_\perp^2 r_H^2$. Thus, in the general case $\gamma = \min(\gamma_1, \gamma_2) > \omega_*$. For $k_\perp \sim \kappa$ eqn. (IV.97) can be used if

$$r_H \kappa < \left(\frac{m_e}{m_i}\right)^{\frac{1}{6}} (\lambda_e \kappa)^{\frac{1}{3}} \left(\frac{v_e}{u}\right)^{\frac{2}{3}} \tag{IV. 99}$$

(h) *Transition to Collisionless Dissipation*

To conclude our discussion of the drift-dissipative instability, we shall briefly consider the transition from dissipation due to collisions to collisionless dissipation (90). Using the simplified expression for the collision term, assuming that after scattering at the neutrals the electrons have a Maxwellian distribution, the linearised kinetic equation for the electrons can be written down in the following form

$$(-i\omega + ik_z v_z) f_{\mathbf{k}\omega} + \frac{e}{m_e} ik_z \varphi_{\mathbf{k}\omega} \frac{\partial f}{\partial v_z} - i\frac{c}{H} k_y \frac{\partial f}{\partial x} \varphi_{\mathbf{k}\omega} = -\nu_e f_{\mathbf{k}\omega} + \nu_e \frac{n_e}{n} f \tag{IV. 100}$$

where $\nu_e = \tau_e^{-1}$, and from this we obtain the electron density perturbation

$$\frac{n_e}{n} = \left\{1 - \frac{\omega - \omega_*}{k_z v_e} Y\left(\frac{\omega + i\nu_e}{k_z v_e}\right)\left[1 - \frac{i\nu_e}{k_z v_e} Y\left(\frac{\omega + i\nu_e}{k_z v_e}\right)\right]^{-1}\right\} \frac{e\varphi}{T_e} \tag{IV.101}$$

where the function $Y(z)$ is defined by eqn. (IV.49).

For $\nu_e \to 0$ we recover the result previously obtained for a collisionless plasma. In the other limiting case $\tau_e k_z v_e = \lambda_z k_z \ll 1$ and $\omega\tau_e \ll 1$ we obtain the hydrodynamic result (IV.85). The transition from collisional to collisionless dissipation occurs broadly speaking at $\lambda_e k_z = 1$. A more exact treatment of this transition (90) shows that oscillations with $\omega \sim \nu_e$ lead to some broadening of the region of the ion-sound instability. This additional region is marked on Fig. 23 by the oblique hatching.

At very small k_z values the function Y can be expanded in inverse powers of the argument. For $\omega\tau_e > 1$, retaining the first two terms of this expansion leads to the expression

$$\frac{n_e}{n} = \left\{1 - \frac{\omega - \omega_*}{\omega}\left[1 + \frac{T_e k_z^2}{\omega(\omega + i\nu_e) m_e}\right]\right\} \frac{e\varphi}{T_e} \tag{IV.102}$$

Comparing this expression with the ion density perturbation which, for drift waves with $T_e = T_i$ and $\omega/k_z \gg v_i$, is given by

$$\frac{n_i}{n} = \left(\frac{\omega_*}{\omega} - s\frac{\omega_* + \omega}{\omega}\right)\frac{e\varphi}{T}$$

according to eqn. (IV.66), we obtain the dispersion relation

$$\frac{\omega-\omega_*}{\omega+\omega_*}=\frac{\omega(\omega+i\nu_e)}{\omega_z^2} \tag{IV. 103}$$

where $\omega_z^2=\frac{k_z^2}{k_\perp^2}\Omega_i\Omega_e$, $\nu_e=\tau_e^{-1}$. We have found earlier (see eqn. (IV.57)) that in this equation complex roots appear for $\nu_e = 0$ and $\frac{\omega_*}{\omega_z} > 0{\cdot}4$. In the presence of collisions the region of instability expands towards large k_z values, when $\omega_z \gg \omega_*$, and we have approximately

$$\omega \cong \omega_*+2i\nu_e\frac{\omega_*^2}{\omega_z^2} \tag{IV. 104}$$

This expression is similar to eqn. (IV.92) which was obtained in the hydrodynamic approximation, and for $T_i = 0$ it can be shown that the factor 2 in the second term of (IV.104) disappears and the two expressions are then identical.

(i) *Current-convective (Spiral) Instability*

We shall now show that the instability of the positive column described in (I.2.b) transforms into the drift instability as the collision frequency decreases.

Consider the simplest case $\Omega_i\tau_i \gg 1$. We can then use eqn. (IV.89) for the perturbation of the ion density and (IV.94) for that of the electron density. Comparing n_i and n_e we obtain the dispersion equation

$$\omega^2+i\omega(\nu_i+D_ek_z^2+\omega_s+ik_zu)-\nu_i(D_ek_z^2+ik_zu)-i\omega_*\left(\omega_s+\frac{ik_zu}{k_\perp^2r_H^2}\right)=0 \tag{IV. 105}$$

For $u = 0$ this equation goes over into (IV.90) and for $\nu_i = 0$ it coincides with (IV.95). For $r_H\kappa \ll 1$ the term $k_z^2D_e$ can be neglected compared with ω_s in the second term of (IV.105) as well as in (IV.90). In addition, for $k_\perp r_H \ll 1$ we can neglect k_zu compared with ω_s in the second term, and in the last term ω_s compared with $k_zu/k_\perp^2r_H^2$. In this approximation for $\omega \ll \omega_s$ we obtain from (IV.105)

$$\omega=\frac{k_zu_0\nu_i}{\omega_s+\nu_i}+i\frac{\omega_*k_zu/k_\perp^2r_H^2-\nu_iD_ek_z^2}{\omega_s+\nu_i} \tag{IV. 106}$$

For sufficiently small k_z values the first term in the expression for the imaginary part of the frequency is larger than the second, and the perturbations increase with time. This constitutes the current-convective instability which originates from the gradient of conductivity across the magnetic field. According to (IV.106) the expression for the growth rate can be represented in the form

$$\gamma=b_iE\sqrt{\frac{b_e}{b_i}}\frac{d\ln n}{dx}\frac{k_y}{k_\perp}\frac{\omega_s}{\omega_s+\nu_i}-D_e\frac{b_i}{b_e}\frac{k_\perp^2}{\Omega_i^2\tau_i^2}\frac{\omega_s}{\omega_s+\nu_i} \tag{IV. 107}$$

where b_i is the ion and b_e the electron mobility, E the longitudinal electric field, related to u by $u = b_e E$. For perturbations with not very large $k_\perp$ values, the second term in (IV.107) can be neglected, and the first attains a maximum at $\omega_s = \nu_i$ given by

$$\gamma = U \frac{d \ln n}{dx} \tag{IV. 108}$$

where

$$U = \tfrac{1}{2} b_i E \sqrt{\frac{b_e}{b_i}} \tag{IV. 109}$$

(j) *Non-linear Drift Flows*

If the conditions for the drift instability are fulfilled, small perturbations will increase with time until the non-linear interactions come into play. To help to visualise at least qualitatively the character of these non-linear oscillations, we can reduce the equations for the non-linear motion to a single equation of the hydrodynamic type. We shall discuss here the derivation of equations of this type for non-linear flows corresponding to the drift-dissipative, current-convective and flute instabilities.

Let us start with the drift-dissipative instability in a currentless plasma. For simplicity we neglect the collisions between the ions and the neutral gas atoms, and assume that $T_i = 0$. In these oscillations with a characteristic scale parallel to the magnetic field considerably larger than that across the field, the longitudinal motion of the ions can be neglected. The transverse motion is principally governed by the electric drift $\mathbf{v} = \frac{c}{H}[\mathbf{h}\nabla\varphi]$, with a small correction due to inertial effects. In the continuity equation this correction can be neglected, and using the incompressibility of the drift velocity, div $\mathbf{v} = 0$, we obtain

$$\frac{\partial n}{\partial t} + \mathbf{v}\nabla n = 0 \tag{IV. 110}$$

Furthermore, summing the electron and ion equations of motion, we obtain the hydrodynamic equation

$$m_i n \frac{d\mathbf{v}}{dt} + \nabla p = \frac{1}{c}[\mathbf{jH}] \tag{IV. 111}$$

where j is the electric current density and p the plasma pressure.

Applying the operation $(\mathrm{curl})_z$ to eqn. (IV.111), we obtain on the right hand side the expression $-\frac{H}{c}\operatorname{div}\mathbf{j}_\perp$, which because of div $\mathbf{j} = 0$ equals $\frac{H}{c}\frac{\partial j_z}{\partial z}$. The value of j_z can be determined from the electron equation of motion, neglecting the longitudinal ion current, giving

$$j_z = -\frac{e^2 n \tau_e}{m_e}\left(\frac{\partial\varphi}{\partial z} - \frac{T}{e}\frac{\partial \ln n}{\partial z}\right) \tag{IV. 112}$$

We thus obtain

$$\operatorname{curl}_z\left(m_i n \frac{d\mathbf{v}}{dt}\right) = -\frac{\partial}{\partial z}\Omega_e \tau_e n \frac{\partial}{\partial z}(e\varphi - T\ln n) \tag{IV. 113}$$

Equations (IV.110), (IV.113) together with the relation $\mathbf{v} = \dfrac{c[\mathbf{h}\nabla\varphi]}{H}$ are then the required equations of non-linear motion. From the hydrodynamic point of view, the potential φ can be considered as a function of the current of two-dimensional motion ($v_z = 0$). Equation (IV.110) describes the variation of density in each plane z = const, and the right hand side of eqn. (IV.113) establishes the connection between the flows in the different planes. It is easily verified that in the linear approximation these equations lead to the dispersion eqn. (IV.91).

In the special case of a helical flow where all quantities are functions of only two variables, namely the distance r from the symmetry axis and $\zeta = \theta - kz$ where θ is the aximuthal angle, eqn. (IV.113) can be integrated once, giving

$$m_i n \frac{d\mathbf{v}}{dt} + \nabla p_* = -\frac{H^2\sigma}{c^2} r^2 k^2 \mathbf{e}_r\left(v_r - \frac{cT}{eHr}\frac{\partial}{\partial\zeta}\ln n\right) \tag{IV. 114}$$

where $\sigma = \dfrac{e^2 n\tau_e}{m_e}$ is the conductivity of the plasma, $\mathbf{e}_r$ the unit vector directed along the radius and p_* an arbitrary function of the co-ordinates. Equation (IV.114) together with the continuity eqn. (IV.110) and the incompressibility condition

$$\operatorname{div}\mathbf{v} = 0 \tag{IV. 115}$$

describes a hydrodynamic helical flow during which the fluid is subjected to an additional force determined by the right hand side of eqn. (IV.114). The second term in the expression for this force leads to the instability.

We can similarly obtain equations for the non-linear flow corresponding to the current-convective instability. To this end we need only replace the inertia term by the frictional force $m_i n\mathbf{v}\nu_i$ in eqn. (IV.111) and add to $-\dfrac{\partial\varphi}{dz}$ the external electric field E in the expression for i. The result is to replace (IV.113) by

$$\operatorname{curl}_z(m_i n\mathbf{v}\nu_i) = -\frac{\partial}{\partial z}\Omega_e\tau_e n\left(e\frac{\partial\varphi}{\partial t} - eE - T\frac{\partial}{\partial z}\ln n\right) \tag{IV. 116}$$

and for the helical flow we then obtain

$$m_i n\nu_i\mathbf{v} + \nabla p_* = -\frac{H^2\sigma}{c^2}r^2k^2\mathbf{e}_r\left(v_r - \frac{cE}{Hrk} - \frac{Tc}{eHr}\frac{\partial}{\partial\zeta}\ln n\right) \tag{IV. 117}$$

For small k the dominant term is the second term in brackets, which for $E/k > 0$ corresponds to a force acting in the radial direction. Neglecting the other terms, we obtain the equation of motion for an inhomogeneous

incompressible fluid in a porous medium in the presence of a radial gravitational force.

Finally let us consider the case of a flow corresponding to the flute instability including the effect of the finite Larmor radius. For simplicity we again replace the diamagnetic expulsive force by an effective gravitational force with acceleration g. Since the velocity of the electrons and of the ions is again determined mainly by the electric drift $\mathbf{v} = \frac{c[\mathbf{h}\nabla\varphi]}{H}$, we again have a two-dimensional incompressible flow. This flow can be described by the magnetohydrodynamic equations including a collisionless " oblique " viscosity (see for instance (99)). It has been shown in (100), (101), that in the linear approximation these equations lead to exactly the same results as are obtained from the kinetic equation for $k_\perp \rho_i \ll 1$.

For $\Omega_i \tau_i \gg 1$ the equation of motion assumes the following form

$$m_i n \frac{d\mathbf{v}}{dt} + \nabla p_* = m_i n \mathbf{g} + \mathbf{F} \tag{IV. 118}$$

where the force $\mathbf{F}$ is given by the expression

$$\mathbf{F} = \frac{cT}{H\Omega_i}(\nabla n\,\Delta\varphi - (\nabla n \nabla)\nabla\varphi) = \frac{T}{\Omega_i}\{\nabla n \operatorname{curl}_z \mathbf{v} + (\nabla n \nabla)[\mathbf{h}\mathbf{v}]_z\} \tag{IV. 119}$$

and the effective pressure p_* is given by $p_* = 2nT + \frac{H^2}{8\pi} + nT\frac{\operatorname{curl}\mathbf{v}}{\Omega_i}$. For the incompressible flow considered here, p_* can be taken to be an arbitrary function of the co-ordinates r, θ.

Now it is the force $\mathbf{F}$ which leads to the stabilisation of small flute perturbations. However, there need not be such a stabilisation for finite perturbations. For it is obvious that an isolated plasma tube in a vacuum, at a point where $g \neq 0$, must be ejected in the direction of $\mathbf{g}$. Such a motion is similar to that occurring in the $m = 1$ mode; the electric field is constant over the tube cross-section and there is no difference between the motion of the electrons and of the ions. One might think that such a tube cannot be in equilibrium when it is immersed in a plasma of considerably smaller (or considerably larger) density. In fact we assume the opposite, that a steady state flow is set up in the plasma which does not lead to any flow towards the walls. Such a flow is stationary in some moving system of co-ordinates. In this co-ordinate system the equipotential surfaces φ = const must coincide with surfaces of constant density n = const, since $(\mathbf{v}\nabla n) = 0$. Considering an individual plasma tube bounded by a surface n = const, we can show that the integral of the force $\mathbf{F}$ over the volume of this tube vanishes. For instance, for the integral of F_x we obtain by integration by parts

$$\int F_x\, d\mathbf{r} = \frac{cT}{H\Omega_i}\oint_s n\left(\frac{\partial^2\varphi}{\partial x\,\partial y}\,dx + \frac{\partial^2\varphi}{\partial y^2}\,dy\right) \tag{IV. 120}$$

where the integration on the right is taken over the surface n = const.

Therefore we can remove n from the integral, and obtain the integral over a closed path of the complete differential $d\left(\frac{\partial\varphi}{\partial y}\right)$, which must vanish. Similarly we can show that the integral of F_y also vanishes. Thus the total additional force acting on a separate tube in a plasma bounded by a surface disappears. It follows, therefore, that such a tube can only maintain itself in equilibrium for certain special flows when the pressure p_* is distributed over its surface so that it exactly balances the frictional force.

From these considerations it would seem that for sufficiently strong perturbations, when plasma tubes become separated from the bulk plasma, and the surface $n = \text{const}$ becomes multiply connected, stabilisation due to finite Larmor radius may be absent.

4. Turbulent Diffusion of a Plasma

Since the only reason for the drift instability is the inhomogeneity of the plasma, oscillations developing in consequence of the instability cannot die away until the inhomogeneity is completely destroyed. In other words, such an instability must lead to turbulent diffusion.

The magnitude of the " diffusion " flux brought about by the oscillations so developed can be determined from the following considerations (103, 102, 212). As we established earlier, in the case of a drift instability ($r_H\kappa \ll 1$) the ions move across the magnetic field mainly due to the electric drift

$$\mathbf{v} = \frac{c[\mathbf{h}\nabla\varphi]}{H}$$

In a homogeneous magnetic field the drift motion is incompressible, $\operatorname{div}\mathbf{v} = 0$, and in the presence of drift oscillations the plasma moves across the magnetic field without additional compression or dilation, i.e. turbulent convection takes place. In this case the displacement of the plasma by a distance ξ leads to a perturbation of the density $n' = \xi\kappa n$. In other words the density fluctuations are determined by oscillations of the displacement ξ. If these oscillations vary harmonically, they do not lead to any net flow of the plasma when averaged over time. On the other hand, if the amplitude of the oscillations increases with time, then each succeeding half-period of the oscillations leads to a slightly greater displacement of the plasma than the previous one, and as a result a mean plasma flux $q = \langle\xi n'\rangle = \gamma\kappa n\langle\xi^2\rangle$ occurs. To order of magnitude this relationship between the flux q and the displacement of the plasma ξ or the density perturbation n' can be retained when the interaction between waves comes into play, since even in the presence of an interaction the characteristic rate of growth of individual wave packets and of their transformation into other packets is of the order of the linear growth rate γ.

Consequently, to determine q, we need to know $\langle\xi^2\rangle$ or $\langle n'^2\rangle$. For $\gamma \sim \omega$, when strong turbulence develops, the oscillation amplitude increases to such an extent that the perturbation of the density

gradient $k_\perp n'$ becomes of the order of the mean gradient, i.e. $k_\perp n' \sim \kappa n$. In this case, therefore, the steady state intensity of the density oscillations with wave number k is of the order of magnitude $|n_k'|^2 \sim \frac{\kappa^2 n^2}{k_\perp^2}$. Consequently

$$q = \frac{\gamma}{\kappa n}\langle n'^2\rangle = \left\langle\frac{\gamma}{k_\perp^2}\right\rangle \kappa n = -\left\langle\frac{\gamma}{k_\perp^2}\right\rangle\frac{dn}{dx}$$

For weak turbulence when $\gamma/\omega \ll 1$, the kinetic wave equation can be used to evaluate the oscillation intensity. If the dispersion relation is of the decay type, that is if the condition $\omega_{\mathrm{k}} = \omega_{\mathrm{k}'} - \omega_{\mathrm{k}-\mathrm{k}'}$ can be satisfied, not identically but at some surface in wave number space, we have to order of magnitude $\gamma < n'^2 > = V^2 < n'^2 >^2$. In this relation the left hand side describes the development of the waves due to the instability, and the right hand side represents the effect of the interaction between the waves. The square of the matrix element can be estimated as $V^2 \sim \omega\frac{k_\perp^2}{\kappa^2 n^2}$, since for $n' \sim \frac{\kappa n}{k_\perp}$, when the gradient of the density perturbation is of the order of the mean gradient, the interaction between the oscillations ought to lead to a damping rate of the order of the frequency. Thus for weak turbulence the intensity of the oscillations must decrease as the increment decreases, and to order of magnitude the quantity $\langle n'^2\rangle \sim \frac{\gamma}{\omega}\frac{\kappa^2}{k_\perp^2}n^2$. Approximately, therefore, $q = -\left\langle\frac{\gamma^2}{\omega k_\perp^2}\right\rangle\frac{dn}{dx}$. The factor multiplying the density gradient in this relation represents the turbulent diffusion coefficient. To estimate the value of the diffusion coefficient, we can use for $k_\perp$ the value for which the diffusion coefficient is a maximum, and write to order of magnitude

$$D \sim \left(\frac{\gamma_k^2}{\omega_k k_\perp^2}\right)_{\max} \qquad \text{(IV. 121)}$$

A similar relationship holds good for the ion sound instability ($r_H\kappa \gg 1$), since analogous considerations can be applied to the electron drift motion across the magnetic field. Indeed the case $r_H\kappa \gg 1$ seems simpler since both in the absence and in the presence of collisions $\gamma \sim \omega \sim kc_s$, and consequently

$$D \sim c_s/\kappa$$

In the presence of a strong magnetic field, when $\kappa r_H \ll 1$, the turbulent diffusion is more sensitive to collisions, and more detailed consideration is therefore necessary and is given below.

(a) *Rarefied Plasma in the Absence of a Longitudinal Current*

Let us consider a plasma of such a low density that collisions can be neglected. More precisely, we assume that collisions on the one hand are so rare that they have no effect on the frequency and growth rate of the

oscillations, whilst on the other hand they occur sufficiently frequently to maintain the Maxwellian velocity distribution of the particles. The coefficient of diffusion corresponding to these assumptions will be designated by D_s.

To evaluate D_s we use results obtained in Section 2(c). Let us first consider a plasma column of unlimited length. For $\beta = \frac{8\pi p}{H^2} > \frac{m_e}{m_i}$ only perturbations with $k\rho_i \gtrsim 1$ are unstable. As a function of k_z the growth rate reaches a maximum at $k_z \sim \kappa\sqrt{\beta}$. In this case $\omega \sim k\kappa\rho_i v_i$, and

$$\gamma/\omega \sim \frac{c_A}{v_e} = \sqrt{\frac{m_e}{m_i \beta}}$$

(we assume $T_i = T_e$). It can be shown (see below (f)) that the main contribution to the diffusion comes from oscillations with $k_\perp \rho_i \sim 1$, and in this case

$$D_s \sim \frac{m_e}{m_i \beta} \rho_i^2 v_i \kappa \sim \frac{m_e}{m_i \beta} \kappa \rho_i D_B \tag{IV. 122}$$

where $D_B \sim \rho_i v_i$ is Bohm's coefficient of diffusion.

Thus in this range of β the diffusion coefficient decreases as β increases, and for $\beta \gtrsim 10^{-1}$ the plasma becomes stable and D_s vanishes (leaving only the classical diffusion due to collisions).

For $\beta < \frac{m_e}{m_i}$ we may have, in addition to the short wave instability with $k_\perp \rho_i \sim 1$, a "hydrodynamic" instability for perturbations with large wavelengths for $\omega_*^2 = \omega_z^2$ (see (IV.57)), and these perturbations make the greatest contribution to the diffusion coefficient. Since $\omega \sim \omega_* < c_A k_z$, the minimum permissible value of $k_\perp$ is $\sim \omega_0/c$ and we obtain, using (IV.121), the corresponding diffusion coefficient

$$D_s \sim \frac{c\kappa}{\omega_0} D_B \sim \sqrt{\frac{m_e}{m_i \beta}} \kappa \rho_i D_B \tag{IV. 123}$$

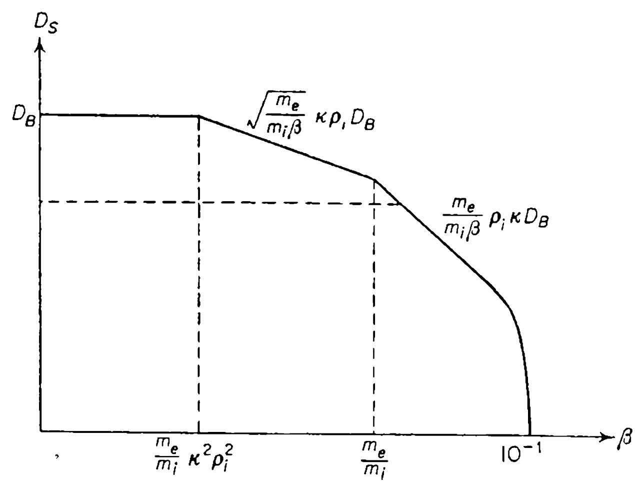

FIG. 24. Coefficient of turbulent diffusion in a tube of infinite length

In this region of magnetic field the coefficient of diffusion is inversely proportional to H, but in absolute value it is smaller than Bohm's coefficient. As the plasma density decreases, when ω_0 becomes of the order $c\kappa$, i.e. for $\beta \sim \frac{m_e}{m_i}\kappa^2\rho_i^2$, the diffusion coefficient reaches its maximum possible value of the order D_B, and does not change when the density further decreases. In this case $\gamma \sim \omega \sim \omega_*$, and the maximum scale of the turbulent pulsations is determined by the tube radius, i.e. $k_\perp \sim \kappa$.

The relationship between D_s and β for an infinitely long column is shown schematically in Fig. 24 by the full line (on a double logarithmic scale).

Now let us examine the effect of shortening the plasma column on the diffusion. The value of the wave number k_{zm}, at which the growth rate reaches its maximum, increases with β, and is given by

$$k_{zm} \sim \kappa\sqrt{\frac{m_e}{m_i}\kappa^2\rho_i^2} \quad \text{for} \quad \beta < \frac{m_e}{m_i}\kappa^2\rho_i^2$$

$$k_{zm} \sim \kappa\beta\sqrt{\frac{m_i}{m_e}} \quad \text{for} \quad \frac{m_e}{m_i}\kappa^2\rho_i^2 < \beta < \frac{m_e}{m_i}$$

and

$$k_{zm} \sim \kappa\sqrt{\beta} \quad \text{for} \quad \beta > \frac{m_e}{m_i}$$

Therefore the effect of finite length of the plasma column is more important for low density plasma. This effect starts at $L < a\sqrt{\frac{m_i}{m_e}}(\kappa\rho_i)^{-2}$ where L is the length of the column and a its radius. For smaller values of L the diffusion coefficient decreases from its value for infinite length, and is given by

$$D_s \sim \left(\frac{m_e}{m_i}\right)^{\frac{1}{4}}\left(\frac{\kappa}{k_0}\right)^{\frac{1}{2}}\kappa\rho_i D_B \quad \text{for} \quad \frac{a}{L} \sim \frac{k_0}{\kappa} < \sqrt{\frac{m_e}{m_i}} \qquad \text{(IV. 124)}$$

and

$$D_s \sim \frac{m_i}{m_e}\left(\frac{\kappa}{k_0}\right)^2\kappa\rho_i D_B \quad \text{for} \quad \frac{a}{L} > \sqrt{\frac{m_e}{m_i}} \qquad \text{(IV. 125)}$$

where $k_0 = 2\pi/L$ is the minimum longitudinal wave number.

Thus, in a tube of infinite length, as β decreases the diffusion coefficient is at first equal to the value for the infinitely long column, but when the value of k_{zm} reaches k_0, the diffusion coefficient remains constant and does not increase as β decreases further. This is shown in Fig. 24 by the " dashed " line.

According to (IV.124), for $\frac{a}{L} < \sqrt{\frac{m_e}{m_i}}$ the coefficient depends relatively weakly on the length of the device: for $\frac{a}{L} \sim \sqrt{\frac{m_e}{m_i}}$ and small β values the

coefficient of diffusion is equal in order of magnitude to $D_s \sim \kappa\rho_i^2 v_i$; for $\frac{a}{L} > \sqrt{\frac{m_e}{m_i}}$ the coefficient D_s decreases fairly rapidly as L decreases, and for $\frac{a}{L} \gtrsim 10^{-1}$ complete stabilisation occurs in general (we do not consider perturbations with $k_z = 0$ which are sensitive to the boundary conditions at the ends of the tube).

(b) *Low Density Plasma carrying a Longitudinal Current*

Consider a plasma column of limited length, say $\frac{a}{L} \sim \sqrt{\frac{m_e}{m_i}}$. We have shown earlier that in the absence of a longitudinal current only short wave oscillations with $k_\perp \rho_i \sim 1$ are excited in the plasma. The corresponding diffusion coefficient is comparatively small, namely $D_s \sim \kappa\rho_i D_B$. In the presence of a longitudinal current, as has been shown in Section 2(b), an additional instability appears at long wave perturbations with $k_\perp \rho_i \ll 1$. Since the diffusion coefficient (IV.121) is proportional to the square of the transverse wavelength, this instability increases its magnitude considerably.

Let us once more consider a plane inhomogeneous plasma layer with density gradient along the x axis and assume that $T_i = T_e = \text{const}$. In equilibrium or, more precisely, in a state averaged over time, the electric field in laboratory co-ordinates, in which the ions are at rest, is determined by the density gradient

$$\nabla\varphi = -\frac{T}{en}\nabla n \tag{IV. 126}$$

The longitudinal electron current excites oscillations of the potential φ' and density n'. According to Section 3(b), waves with longitudinal phase velocity considerably smaller than the electron thermal velocity are excited in the plasma and for these waves all electrons except the resonance electrons can attain equilibrium parallel to the magnetic field. Therefore the perturbation of the potential can be represented in the form

$$\varphi' = \frac{T}{e}\ln\left(1 + \frac{n'}{n}\right) + \varphi_1 \tag{IV. 127}$$

where the first term represents the Boltzmann distribution, and the small term φ_1 is related to the resonance electrons.

If the Larmor radius of the ions is appreciably smaller than the characteristic wavelength of the perturbations considered, the hydrodynamic approximation can be used for the ions. In this approximation it is sufficient to consider only the electric drift in the continuity equation, since the Larmor current does not lead to a change of density

$$\frac{\partial n'}{\partial t} + \frac{c}{H}[\mathbf{h}\nabla(\varphi+\varphi')]\,\nabla(n+n') = 0 \tag{IV. 128}$$

Using relation (IV.127), this equation can be written in the form

$$\frac{\partial n'}{\partial t} - \frac{2cT}{eHn}[\mathbf{h}\nabla n]\nabla n' + \frac{c}{H}[\mathbf{h}\nabla\varphi_1]\nabla(n+n') = 0 \qquad \text{(IV. 129)}$$

For $\varphi_1 = 0$ this equation is linear in n'. This circumstance, which indicates that there is no interaction between the modes for a Boltzmann distribution of the electrons along the lines of force, is very important for all that follows.

In eqn. (IV.129) the second term describes the transfer of the perturbation along the y axis. For $\frac{1}{n}\frac{dn}{dx} = \text{const}$ the velocity of this transfer

$$v_0 = -\frac{2cT}{eHn}\frac{dn}{dx}$$

is also constant, and by transforming to a moving co-ordinate system, the second term in (IV.129) could be eliminated. However, for a general distribution of the mean density $n(x)$ this velocity is also a function of x, and because of the resulting differential motion the density perturbation is deformed with time. To demonstrate the character of this deformation, we shall go over to a spectral representation. We put $n' = \int n_{\mathbf{k}\omega} e^{-i\omega t + i\mathbf{kr}} d\mathbf{k}\, d\omega$, where $n_{\mathbf{k}\omega}(\mathbf{r}, t)$ is the slowly varying amplitude of the wave packet $\mathbf{k}$, ω. For $k \gg \kappa$, so that the wavelength of the perturbation is considerably smaller than the transverse dimension a, the quantity $\frac{1}{n}\frac{dn}{dx}$ can be expanded in series near the point $x = x_0$ under consideration, and we shall retain the first two terms of this series. Since in transforming to the Fourier representation $x - x_0 \to i\frac{\partial}{\partial k_x}$, we obtain

$$-i(\omega - 2\omega_*)n_{\mathbf{k}\omega} + \frac{\partial n_{\mathbf{k}\omega}}{\partial t} + 2\frac{d\omega_*}{dx}\frac{\partial n_{\mathbf{k}\omega}}{\partial k_x} - \sqrt{\pi}\frac{u}{v_e}\omega_* n_{\mathbf{k}\omega} -$$
$$-\frac{\sqrt{\pi}}{n}\frac{u}{v_e}\int \omega'_* n_{\mathbf{k}'\omega'} n_{\mathbf{k}-\mathbf{k}',\omega-\omega'} d\mathbf{k}'\, d\omega' = 0 \qquad \text{(IV. 130)}$$

where $\omega_* = k_y v_0 = -\frac{cTk_y}{eHn}\frac{dn}{dx}$, and u is the longitudinal mean electron velocity.

In eqn. (IV.130) the last two terms, the first of which describes the build-up of the oscillations by the resonance electrons, and the second the non-linear interaction between the waves, come from the Fourier transform of the last term in (IV.129). For φ_1 we insert its approximate value

$$\varphi_{1\mathbf{k}\omega} = i\sqrt{\pi}\frac{T}{e}\frac{u}{v_e}\frac{n_{\mathbf{k}\omega}}{n} \qquad \text{(IV. 131)}$$

which can be obtained from the expression for the density perturbation

(IV.50) for $\omega/k_z \ll v_e$. Strictly speaking, the expression (IV.131) should contain an additional factor $k_z/|k_z|$, which we have omitted, assuming $k_z > 0$.

In eqn. (IV.130) the first term is considerably larger than the rest, and therefore as a first approximation we obtain $\omega = 2\omega_*$. In other words, the function $n_{\mathbf{k}\omega}$ is close to $n_{\mathbf{k}}\delta(\omega - 2\omega_*)$. The remaining small terms in (IV.130) describe the evolution of the wave packet in time. In particular, the term with $\frac{\partial n_{\mathbf{k}\omega}}{\partial k_x}$ shows that the deformation of the wave packet due to the differential drift velocity v_0 leads to an increase of the wave number k_x with time, i.e. to a flow in wave number space. To order of magnitude this term is equal to

$$\omega_* \frac{\kappa}{k_\perp} n_{\mathbf{k}\omega}$$

and for small $k_\perp$ in particular it will be the decisive term: all perturbations will " drift " into the region of large k_x faster than their amplitude grows due to the instability. For $\frac{k_\perp}{\kappa} > \frac{v_e}{u}$ the instability begins to be dominant, and the distortion of the waves can be neglected. Neglecting the second term in (IV.130) and introducing the new variable $\nu = \omega - 2\omega_*$, we obtain for the region $\kappa/k_\perp < u/v_e$:

$$\nu n_{\mathbf{k}\nu} - i\sqrt{\pi}\,\omega_* \frac{u}{v_e} n_{\mathbf{k}\nu} - i\frac{\sqrt{\pi}}{n}\frac{u}{v_e}\int \omega'_* n_{\mathbf{k}'\nu'} n_{\mathbf{k}''\nu''}\, d\mathbf{k}'\, d\nu' \qquad \text{(IV. 132)}$$

where $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$ and $\nu'' = \nu - \nu'$. (We neglect here the non-linear Landau damping at the ions, assuming $u \gg v_i$.)

In the linear approximation, it is clear from eqn. (IV.132) that the characteristic frequency ν is purely imaginary, so that we are concerned with an aperiodic instability which must lead to strong turbulence. This can also be seen directly from eqn. (IV.132) according to which the non-linear interaction can balance the linear growth of the oscillations only when the perturbation of the density becomes of the order of the mean density. This result is a direct consequence of the absence of interaction between modes, noted above, for a Boltzmann distribution of the electrons.

Let us estimate the value of the diffusion flux $q = \left\langle \frac{cE_y}{H} n' \right\rangle$. We must substitute $-\frac{\partial \varphi_1}{\partial y}$ for E_y, since only the resonant electrons undergo diffusion (it is easily seen that the term containing $\frac{\partial}{\partial y}\frac{T}{e}\ln\left(1 + \frac{n'}{n}\right)$ vanishes when averaged against n'). Using (IV.131) we obtain

$$q = \frac{cT}{eH}\sqrt{\pi}\frac{u}{v_e}\int \frac{k_y k_z}{|k_z|}\frac{N_{\mathbf{k}\omega}}{n} d\mathbf{k}\, d\omega \qquad \text{(IV. 133)}$$

where $N_{\mathbf{k}\omega}$ is the spectral function of the density:

$$\langle n_{\mathbf{k}\omega} n_{\mathbf{k}'\omega'}\rangle = N_{\mathbf{k}\omega}\,\delta(\omega-\omega')\,\delta(\mathbf{k}-\mathbf{k}').$$

In the case of strong turbulence the density perturbation is of the order of the mean density and consequently to order of magnitude $q \sim \frac{cT}{eH}\frac{u}{v_e}k_\perp n$, where $k_\perp$ is a typical value of the wave number in the region of strong turbulence. As we have shown above, this region begins at $k_\perp \sim \kappa\frac{u}{v_e}$. Substituting this value of $k_\perp$ into the expression for q (since the effective value of $k_\perp$ is in any case not smaller than this value), we obtain to order of magnitude $q \sim -\frac{cT}{eH}\frac{dn}{dx}$.

Thus, in the presence of a longitudinal current the diffusion coefficient of the plasma in a very strong magnetic field is of the order of the Bohm value.† This result is valid only when the value of $\frac{u}{v_e}$ is not excessively small. In fact, for $\frac{a}{L} \sim \sqrt{\frac{m_e}{m_i}}$, as well as the build-up of the long wave perturbations considered here, there is an instability at perturbations with $k_\perp\rho_i \sim 1$ which also leads to diffusion. Since the diffusion coefficient determined by the short wave perturbations is of the order of $\kappa\rho_i^2 v_i$, the long wave perturbations must be damped at a rate of the order of $\sim k_\perp^2\kappa\rho_i^2 v_i$. Clearly the results given above are only valid if for perturbations with $k_\perp \sim \kappa\frac{v_e}{u}$ the growth rate $\gamma \sim \frac{u}{v_e}\omega_*$ is larger than this damping rate, i.e.

$$\frac{u}{v_e} > \sqrt{\kappa\rho_i} \tag{IV. 134}$$

and for smaller values of u/v_e the coefficient of diffusion must drop quite sharply to a value $\sim \kappa\rho_i D_B$, since the differential rotation of the plasma $\left(\frac{dv_0}{dx} \neq 0\right)$ shifts the long wave perturbations into the region of large k_x where they are damped by diffusion arising from the short wave perturbations. In the presence of even a small shear of the magnetic field, of the order $\theta \sim \kappa\rho_i$, the short wave oscillations are stabilised, and then the turbulent diffusion coefficient may rise to $\sim D_B$ even for small values of u/v_e.

† In ref. (103) we assumed for the velocity fluctuation $v' \sim \frac{cEy}{H} \sim \frac{\gamma}{\kappa}\frac{n'}{n}$ too low a value, namely $v' \sim \gamma n'/k_\perp n$, so that the result obtained there, $D \sim \frac{u}{v_e} D_B$, is also too low.

(c) *Diffusion of a Very Low Density Plasma*

The results given above are also invalid unless collisions can in fact maintain a Maxwellian distribution of the longitudinal velocity of the electrons. In a very tenuous plasma this may not be possible, and then conclusions based on the specific form of the distribution function may need considerable revision. A decrease in the collision frequency must have a considerable effect on the diffusion either in the presence of a longitudinal flux, or for $\beta > m_e/m_i$, where the oscillations are built up by a small number of resonant electrons. For the case $\beta < m_e/m_i$ on the other hand, where the instability in a long tube is of a hydrodynamic type, and all electrons participate in the growth of the oscillations, the results are not sensitive to reduction of the collision frequency.

Consider first the case of diffusion due to longitudinal current. For the preceding results to remain valid, the distribution function must be similar to a displaced Maxwellian distribution with an accuracy up to u/v_e. On the other hand, the diffusion loss leads to an appreciable distortion of the distribution function in a time of the order of the characteristic diffusion time $\kappa^2 D \sim \kappa^2 D_B$, and this distortion can be destroyed by collisions only if $\nu_e \frac{u}{v_e} > \kappa^2 D_B$. Introducing the parameter $S = \lambda_e \rho_i \kappa^2$, this condition can be written in the form:

$$S < u/v_e \tag{IV. 135}$$

In a very tenuous plasma this condition is infringed, and a plateau develops on the distribution function. The escape time is then determined by the rate at which collisions can restore the Maxwellian distribution, and in order of magnitude this cannot exceed $\frac{u}{v_e}\nu_e$.

Turning to the case of a currentless plasma, for $\beta \gg m_e/m_i$ we also find that only a small number of resonant electrons, with longitudinal velocity $v_z < c_A = v_e\sqrt{\frac{m_e}{m_i \beta}}$ participate in the diffusion. A decrease in the collision frequency then again leads to a decrease in the diffusion coefficient.

(d) *Diffusion of a Dense Plasma*

The expressions given in Section 4(a) above for the turbulent diffusion coefficient D_s refer to a fairly tenuous plasma where the collision frequency ν_e is smaller than $\omega \sim \omega_*$ for all waves including $k_\perp \sim \kappa$. This condition can be written in the form

$$S > \sqrt{\frac{m_i}{m_e}} \tag{IV. 136}$$

For smaller values of the parameter $S = \lambda_e \rho_i \kappa^2$ the friction between the electrons and ions due to collisions becomes important, leading to the

drift-dissipative instability. The collisions lead to an increase in the optimum wave number k_z, at which the growth rate reaches its maximum, and from the condition $\omega_s = \omega_*$ we obtain for $k_\perp = \kappa$

$$k_z \approx \left(\frac{m_e}{m_i}\right)^{\frac{1}{4}} \kappa^2 \rho_i \sqrt{\frac{\rho_i}{\lambda_e}} = \frac{\kappa^2 \rho_i}{\sqrt{\Omega_e \tau_e}} \tag{IV. 137}$$

The longitudinal phase velocity corresponding to this value of k_z is of the order of $\omega/k_z \approx v_i \left(\frac{m_e}{m_i}\right)^{\frac{1}{4}} \left(\frac{\lambda_e}{\rho_i}\right)^{\frac{1}{2}}$, and if this velocity is smaller than the Alfvén velocity c_A, and k_z given by (IV.137) is larger than the minimum possible value $k_0 = 2\pi/L$, then $\gamma \sim \omega \sim \omega_*$ for $k_\perp \sim \kappa$, and the corresponding coefficient of turbulent diffusion attains the value $D_B \sim \rho_i v_i$. According to (IV.137) the condition $\omega/k_z < c_A$ can be written in the form $\beta < (\Omega_e \tau_e)^{-1}$, and if this condition is violated we can have $\gamma \sim \omega$ only for larger values of $k_\perp$. The diffusion coefficient then decreases to

$$D \approx \frac{c_A^2}{\Omega_i \Omega_e \tau_e} = \frac{D_\perp}{\beta} \tag{IV. 138}$$

where $D_\perp = \rho^2/\tau_e$ is the classical coefficient of diffusion. At the same time the optimum wave number increases: $k_z \approx \kappa^2 \rho_i \beta^{\frac{3}{2}} \Omega_e \tau_e$.

In a device of limited length, when the minimum wave number $k_0 = 2\pi/L$ is larger than the optimum values given above, the coefficient of diffusion is determined by shorter wave perturbations, for which again $\gamma \sim \omega$ (the contribution from the long wave perturbations $\sim \gamma^2/\omega k_\perp^2$ varies as $k_\perp^5$ as $k_\perp$ decreases). The corresponding coefficient of diffusion can be estimated as

$$D \approx D_B \left(\frac{\kappa^2 \rho_i}{k_0 \sqrt{\Omega_e \tau_e}}\right)^{\frac{2}{3}} \tag{IV. 139}$$

This expression in its turn is valid only for $k_0/\kappa < \left(\frac{m_e}{m_i}\right)^{\frac{1}{4}} \lambda_e \kappa$, since otherwise the optimum wave number $k_\perp$ becomes so large that viscous damping of the oscillations $\sim k_\perp^2 \rho_i^2/\tau_i$ becomes important. Moreover, expression (IV.139) is valid only for $k_0 \lambda_e < 1$, where the diffusion approximation can be used, while if this expression gives a greater value than (IV.138) the latter must be used.

We shall now consider the effect of a longitudinal current. We have shown above that in a tenuous plasma the diffusion coefficient in the presence of a longitudinal current may reach the value D_B. This conclusion requires $k_z \lambda_e > 1$, so that collisions can be neglected. Since the maximum possible value of k_z is determined by $\omega/k_z \sim \frac{\omega_*}{k_z} \sim v_i$, the condition for the validity of the collisionless approximation is

$$S > 1 \tag{IV. 140}$$

For smaller values of $S = \lambda_e \kappa^2 \rho_l$ the diffusion approximation can be used. According to Section 3(g), in a strong magnetic field $(\kappa^2 \rho_i^2 \ll 1)$ and with a sufficiently large longitudinal current $(u/c_s > \kappa^2 \rho_i^2 \sqrt{\Omega_e \tau_e})$ the growth rate of small perturbations (IV.97) is larger than the oscillation frequency $\sim \omega_*$. The corresponding coefficient of diffusion can be estimated as

$$D \sim \frac{c_s}{\kappa} \left(\frac{u^2}{c_s v_e \lambda_e \kappa} \right)^{\frac{1}{3}} \sim D_B \left(\frac{u}{u_*} \right)^{\frac{2}{3}} \qquad \text{(IV. 141)}$$

where

$$u_* = c_s \kappa^2 r_H^2 \sqrt{\Omega_e \tau_e} \qquad \text{(IV.142)}$$

Relation (IV.141) refers to a plasma of fairly low pressure $(\beta < (\Omega_e \tau_e)^{-1})$ in a long device $\left(k_0 < k_* = \kappa \left(\frac{u c_s \kappa \rho_i^3}{v_e^2 \lambda_c^2} \right)^{\frac{1}{3}} \right)$. As k_0 increases and becomes greater than k_*, the diffusion coefficient varies inversely as k_0 and is given by

$$D \sim D_B \frac{k_*}{k_0} \left(\frac{u_0}{u_*} \right)^{\frac{2}{3}} = D_B \frac{u}{v_e k_0 \lambda_e} \qquad \text{(IV. 143)}$$

This relation can be used as long as the condition $k_0 \lambda_e < 1$ is satisfied, or provided that the value given by (IV.143) is less than that given by (IV.139). According to eqns. (IV.141) and (IV.143), the diffusion coefficient in a long tube carrying a current such that $u > u_*$ is independent of the magnetic field, while for a shorter tube it decreases as H^{-1}.

According to the results of Section 3(g), the longitudinal current may also increase the diffusion coefficient in the case of a weak magnetic field $r_H \kappa > 1$, where an ion-sound instability occurs.

All these estimates for the turbulent diffusion coefficient are very rough, and for systems of limited length they may not be applicable because we have not considered the possibility of perturbations uniform along the length of the system $(k_z = 0)$, which require special consideration for each specific case, since they must be sensitive to the boundary conditions at the end of the tube.

Since turbulence developing from the drift instability is usually strong, to obtain a more accurate determination of the turbulent diffusion coefficient it would be necessary either to use the weak coupling approximation, or to introduce the free mixing length. We shall consider below two special cases for which a more detailed calculation has been carried out, one a case of strong and the other of weak turbulence.

(e) *Turbulent Positive Column* (17)

Consider an infinitely long positive column in an insulating tube in a longitudinal magnetic field, much higher than the critical field. We established in Section 3 that such a column is unstable to perturbations which are greatly

extended along the magnetic field, with a growth rate maximised over k_z given by

$$\gamma = U\frac{d\ln n}{dx} \tag{IV. 144}$$

for $\Omega_i\tau_i \gg 1$, where $U = \frac{1}{2}b_iE\sqrt{\frac{b_e}{b_i}}$. This instability gives rise to a convective flow in the plasma during which the plasma escapes to the wall in the form of separate helical tubes, while outside the plasma isolated " bubbles " appear. Since at the boundary of these " bubbles " the relative gradient $\frac{\nabla n}{n}$ is very large, the instability causes them to move about fairly rapidly with the plasma and the motion assumes a chaotic and turbulent character.

The magnitude of the diffusion flux in such a turbulent plasma can be determined by using the mixing length concept. Suppose l is the effective length or mixing length by which the plasma tubes are displaced by the convective motion, before the coherence of this motion is destroyed due to the interaction with other perturbations. Since the perturbation of the density arises primarily from the convective motion, the density fluctuation level is given by $n' = l\frac{dn}{dx}$. The velocity fluctuation v', related to the density fluctuation by $\gamma n' \sim v'\frac{dn}{dx}$, can be evaluated as $v' = U\frac{n'}{n}$, and consequently the diffusion flux $q = \langle n'v' \rangle$ is given by

$$q = -Ul^2\left|\frac{\nabla n}{n}\right|\nabla n \tag{IV.145}$$

In a discharge in an insulating tube the walls have no stabilising effect on the pulsations, and by analogy with turbulent jets in an ordinary fluid the value of l may be assumed constant over the cross-section and proportional to the radius of the tube a. Using expression (IV.145) it is then straightforward to set up a particle balance equation to determine the diffusion losses and the radical density distribution. The mixing length l can then be determined by comparing the results with experiment. Such a comparison was carried out by the author (17), who obtained the following results for values of the magnetic field at which the turbulent loss rate is equal to the loss rate without a magnetic field: for discharges in helium $l/a = 0{\cdot}15$, in hydrogen $l/a = 0{\cdot}10$, and in nitrogen $l/a = 0{\cdot}12$. Thus just as for the turbulent jet, we have $l/a \sim 0{\cdot}1$.

Figure 25 shows a comparison of the experimentally measured and theoretically calculated dependence on the magnetic field of $\theta \equiv E_s/E_0$, where E_s is the electric field in the turbulent discharge, and E_0 the electric field for $H = 0$. The good agreement between the experimental and theoretical curves shows that the description of the turbulent plasma on the

basis of the mixing length concept is fully justified. Moreover, experimental measurement of the radial density distribution (104) confirms the validity of the assumption l = const. The theoretical and experimental density distributions are compared in Fig. 26. The boundary condition

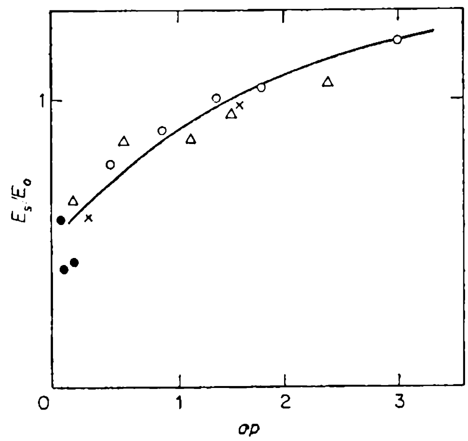

FIG. 25. Dependence of ratio of the longitudinal electric field E_s in a turbulent discharge to the field E_0 for $H = O$ on the neutral gas pressure, for discharges in helium

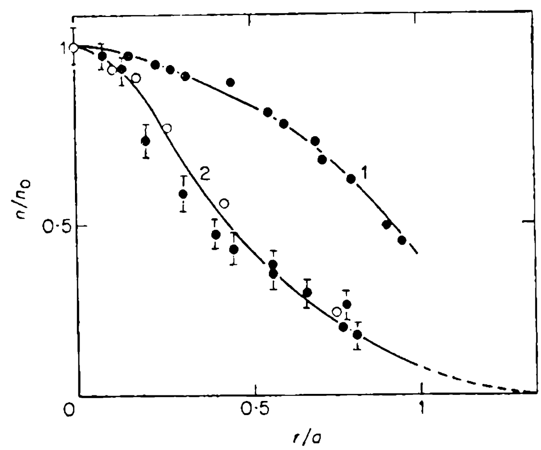

FIG. 26. Comparison of theoretical and experimental radial distribution of plasma density (1) Discharge without magnetic field. (2) Turbulent discharge

$q = n_s U$ was assumed in calculating the theoretical distribution where n_s is the density at the wall. This boundary conditon is analogous to the introduction of the extrapolated length in ordinary diffusion. Figure 26 shows that the actual density distribution is closely similar to the theoretical form and very different from the distribution in the absence of the magnetic field, shown on the same figure.

(f) *Turbulent Diffusion of a Tenuous Plasma with $\beta \gg \frac{m_e}{m_i}$*

The second example to be considered is of interest in that it shows how peculiar may be the interaction between the oscillations in a turbulent plasma.

Consider an inhomogeneous low density collisionless plasma in a strong magnetic field and suppose $\rho_i \kappa \ll 1$, $\beta \gg m_e/m_i$. It was shown in Section 3 that such a plasma is unstable to short-wave perturbations with $k\rho_i \sim 1$. For $\beta \gg m_e/m_i$ the growth rate of perturbations with $k\rho_i \sim 1$ is considerably smaller than the frequency, so that the motion of the plasma is weakly turbulent and can be described by the kinetic wave equation. Such an equation has been set up and analysed in (105) (see also (106)). Only the results of these investigations will be given here.

The dispersion relation of the excited drift waves is of the decay type, and the decay of the waves leads to diffusion of energy in momentum space. In addition to the decay interaction, the non-linear damping of the waves due to beat oscillations with phase velocity $\omega''/k''_z \lesssim v_i$ is important for these waves. The matrix element of the wave interaction is small for modes with wave vectors of similar absolute value, but is appreciable for modes with very different wave numbers. Because of this, perturbations with $k_\perp \rho_i \sim 1$, though not strongly interacting among themselves, can nevertheless completely suppress oscillations with $k_\perp \rho_i \gg 1$, and the spectral function has a distinct maximum at $k_\perp \rho_i \sim 1$. The amplitude of the oscillations of the potential and density near the maximum are determined by the balance between the linear build-up of the oscillations due to resonant electrons and the non-linear damping at the beats, and the shape of the spectrum is determined by the decay interaction. As a result of the suppression of the short-wave perturbations the value of the turbulent diffusion coefficient is determined by (IV.121) not at the maximum of this expression, which is attained for $k_\perp^2 \rho_i^2 \sim m_i/m_e\beta$, but at $k_\perp \rho_i \sim 1$, where the value is

$$D \sim \frac{m_e}{m_i \beta} \rho_i^2 v_i \kappa \tag{IV. 146}$$

The potential fluctuations are of the order of $\varphi' \sim \dfrac{T\kappa}{e}\left(\dfrac{m_e}{m_i\beta}\right)^{\frac{1}{2}}$ and those of the density of the order of $n'/n \sim \kappa\rho_i \left(\dfrac{m_e}{m_i\beta}\right)^{\frac{1}{2}}$. This is to be expected since the perturbation of the density arises from the displacement of the plasma by about a Larmor radius.

(g) *Electrical Conductivity of a Turbulent Plasma*

The classical diffusion of a completely ionised plasma is determined by

the frictional force between the electrons and ions. Including this frictional force the hydrodynamic equations take the form:

$$\nabla p_e = -enE - \frac{en}{c}[\mathbf{vH}] + \frac{1}{c}[\mathbf{jH}] + \frac{en}{\sigma}\mathbf{j} \tag{IV. 147}$$

$$m_i n\frac{d\mathbf{v}}{dt} + \nabla p_i = en\mathbf{E} + \frac{en}{c}[\mathbf{vH}] - \frac{en}{\sigma}\mathbf{j} \tag{IV. 148}$$

where $p_e = nT_e$, $p_i = nT_i$, $\mathbf{j} = en(\mathbf{v}-\mathbf{v}_e)$, $\sigma = \dfrac{e^2 n\tau_e}{m_e}$ is the conductivity, and $\mathbf{v}$ the ion velocity.

From (IV.147), (IV.148) we find that in equilibrium $\mathbf{j}_\perp = \dfrac{c}{H}[\mathbf{h}\nabla p]$, where $p = p_i + p_e$ is the total plasma pressure. If the plasma is inhomogeneous along the x axis and $E_y = 0$, we can determine the diffusion velocity due to collisions from the y component of either of the above equations.

$$v_x = -\frac{c}{H\sigma}j_y = -\frac{1}{m_e\tau_e\Omega_e^2}\frac{dp}{dx} \tag{IV. 149}$$

When oscillations are present in the plasma, we must include in equations (IV.147) and (IV.148) additional terms of the form $\pm\langle en'\mathbf{E}'\rangle$, where n' is the density fluctuation, and $\mathbf{E}'$ that of the electric field. These terms describe turbulent diffusion, and the y component in this expression leads to diffusion along the x axis. Formally they might be treated as an additional " frictional force " representing a decrease in the effective conductivity of the plasma. However, in practice enhanced diffusion cannot generally be attributed to an enhanced transverse resistivity. This would be possible if the reason for the instability were only the difference between the drift velocities, but as we have seen, the instability really arises from the inhomogeneity of the plasma, and the addition of a further velocity difference between the electrons and ions by the imposition of an external field is not generally equivalent to an increase of the density gradient.

Nevertheless, the oscillations may have a considerable influence on the conductivity of the plasma across the magnetic field. It is well known that the effective conductivity of the plasma across a magnetic field depends strongly on the boundary conditions on the Hall current. This effect can be observed directly from eqn. (IV.147) for $\nabla p_e = 0$, $\mathbf{v} = 0$. If there are no constraints on the component of the current perpendicular to the external field $\mathbf{E}_\perp$ (the so-called Hall current), then $\mathbf{j}_\perp = \dfrac{\sigma}{(\Omega_e\tau_e)^2}\mathbf{E}_\perp$, but if the Hall current is prevented from flowing, a Hall electric field, perpendicular to the external field, is set up in the plasma and the normal conductivity $\mathbf{j}_\perp = \sigma\mathbf{E}_\perp$ is restored. Any non-uniformity of the plasma conductivity makes the passage of the Hall current difficult, so that even a small non-uniformity of a strongly magnetised plasma may have a considerable influence

on the flow of current along **E**. This effect has been demonstrated by Yoshikawa and Rose (107).

Consider first the simplest case, of high frequency oscillations where the ions may be considered at rest. For simplicity we put σ/n = const, so that the oscillations of the conductivity are determined by those of the density (corresponding in practice to a weakly ionised plasma). Then the equations for the fluctuations n', $\mathbf{E}'$, $\mathbf{j}'$ and the mean values n, $\mathbf{E}$, $\mathbf{j}$, can be written in the form:

$$T_e \nabla n' = -en'\mathbf{E} - en\mathbf{E}' + \frac{1}{c}[\mathbf{j}'\mathbf{H}] + \frac{en}{\sigma}\mathbf{j}' \qquad \text{(IV. 150)}$$

$$T_e \nabla n = -en\mathbf{E} - e\langle n'\mathbf{E}'\rangle + \frac{1}{c}[\mathbf{j}\mathbf{H}] + \frac{en}{\sigma}\mathbf{j} \qquad \text{(IV. 151)}$$

From (IV. 150) we obtain

$$\mathbf{j}'_\perp = \frac{c}{H^2}[\mathbf{H}, T_e\nabla n' + en'\mathbf{E} - en\nabla\varphi'] + \frac{c^2 en}{H^2\sigma}(T_e\nabla n' + en'\mathbf{E} - en\nabla\varphi') \qquad \text{(IV. 152)}$$

where $\nabla\varphi' = -\mathbf{E}'$.

Transforming to a Fourier representation (assuming $\kappa \ll k_\perp$), and using div $\mathbf{j}' = 0$, we obtain a relationship between the perturbations of the potential $\varphi_\mathbf{k}$ and the density $n_\mathbf{k}$

$$e\varphi_\mathbf{k}(k^2 + \Omega_e^2\tau_e^2 k_z^2 - i\kappa k_y \Omega_e \tau_e)$$
$$= (T_e k^2 + T_e k_z^2 \Omega_e^2 \tau_e^2 - ik_z E_z e\Omega_e^2\tau_e^2 - ik_y E_x e\Omega_e\tau_e)\frac{n_\mathbf{k}}{n} \qquad \text{(IV. 153)}$$

(we assume that $E_y = 0$). Substituting this value for $\varphi_\mathbf{k}$ into the averaged eqn. (IV.151), we obtain an expression for the additional current

$$\delta j_x = j_x - \frac{\sigma E_x}{\Omega_e^2 \tau_e^2}$$

which for $\Omega_e\tau_e \gg 1$ becomes

$$\delta j_x = -\frac{ec}{H}\langle n'E_y'\rangle$$
$$= \sigma\int\left\{E_x + \Omega_e\tau_e\frac{k_z}{k_y}E_z + \frac{T_e}{en}\frac{dn}{dx}\right\}\frac{k_y^2(k^2+\Omega_e^2\tau_e^2k_z^2)N_\mathbf{k}n^{-2}}{(k^2+\Omega_e^2\tau_e^2k_z^2)^2+\kappa^2k_y^2\Omega_e^2\tau_e^2}d\mathbf{k} \qquad \text{(IV. 154)}$$

and similarly

$$\delta j_z = \frac{\sigma}{n}\langle n'E_z'\rangle$$
$$= -\sigma\int\left\{E_x + \Omega_e\tau_e\frac{k_z}{k_y}E_z + \frac{T_e}{en}\frac{dn}{dx}\right\}\frac{k_z k_y\Omega_e\tau_e(k^2+\Omega_e^2\tau_e^2k_z^2)N_\mathbf{k}n^{-2}}{(k^2+\Omega_e^2\tau_e^2k_z^2)^2+\kappa^2k_y^2\Omega_e^2\tau_e^2}d\mathbf{k}$$
$$\text{(IV. 155)}$$

When the inhomogeneity of the mean distribution is small, so that $\kappa\Omega_e\tau_e \ll k^2$, we can neglect the last terms in the denominators of the expressions below the integral sign, and the expressions for the additional

current are then somewhat simpler. Assuming in addition that the spectral function of the density $N_{\mathbf{k}}$ is isotropic, we can perform the angular integration in the above expressions, obtaining the simpler expressions

$$\delta j_x = \frac{\pi\gamma}{4}\frac{\sigma}{\Omega_e\tau_e}\left(E_x + \frac{T_e}{e}\frac{1}{n}\frac{dn}{dx}\right), \qquad \delta j_z = -\gamma\sigma E_z \tag{IV. 156}$$

where $\gamma = \int N_{\mathbf{k}} n^{-2}\, d\mathbf{k} \lesssim 1$.

Thus in the presence of these oscillations the transverse current increases by a factor of almost $\Omega_e\tau_e$. In the case of non-isotropic oscillations the effect may be considerably larger. For instance, in the presence of perturbations which are greatly extended along the magnetic field, so that $k_z \sim \Omega_e^{-1}\tau_e^{-1}$, the transverse conductivity may reach values of the order σ. This effect has been demonstrated experimentally (107).

The effect of the inhomogeneity on the longitudinal conductivity is considerably smaller; the conductivity is only slightly less than the collisional value. Moreover, when the spectral function $N_{\mathbf{k}}$ depends on the direction of the wave vector, the transverse electric field may excite a longitudinal current. This effect is described by the second term in braces in eqn. (IV.155). It occurs only when the perturbations are inclined to the magnetic field on the average, so that $\langle k_z/k_y\rangle \neq 0$. According to (IV.154), the longitudinal field E_z then excites a transverse current. This effect is analogous to the appearance of an averaged particle flux in a turbulent positive column where the mean drift originates from the longitudinal electric field.

The above results refer to high frequency oscillations ($\omega \gg \Omega_i$) where the ions are at rest. For low frequency oscillations the effect may be considerably smaller. In this case we must take into account the perturbation of the ion velocity v' in the equations of motion, which become:

$$T_e\nabla n = -en\mathbf{E} - e\langle n'\mathbf{E}'\rangle - \frac{e}{c}\langle n'[\mathbf{v}'\mathbf{H}]\rangle + \frac{1}{c}[\mathbf{jH}] + \frac{en}{\sigma}\mathbf{j} \tag{IV. 157}$$

$$T_e\nabla n' = -en'\mathbf{E} - en\mathbf{E}' - \frac{e}{c}n[\mathbf{v}'\mathbf{H}] + \frac{1}{c}[\mathbf{j}'\mathbf{H}] + \frac{en}{\sigma}\mathbf{j}' \tag{IV. 158}$$

$$0 = en\mathbf{E} + e\langle n'\mathbf{E}'\rangle + \frac{e}{c}\langle n'[\mathbf{v}'\mathbf{H}]\rangle - \frac{en}{\sigma}\mathbf{j} \tag{IV. 159}$$

$$m_i n\frac{d\mathbf{v}'}{dt} = en\mathbf{E}' + en'\mathbf{E} + \frac{e}{c}n[\mathbf{v}'\mathbf{H}] - \frac{en}{\sigma}\mathbf{j}' \tag{IV. 160}$$

We can determine $\mathbf{j}_\perp$ from the sum of eqns. (IV.158), (IV.160) and then use the condition $\operatorname{div}\mathbf{j} = 0$ together with the longitudinal component of eqn. (IV.158) to obtain

$$\begin{aligned}\operatorname{div}\mathbf{j}'_\perp &= -i\frac{c}{H}\frac{\omega}{\Omega_i}(enk^2\varphi_{\mathbf{k}} - ien_{\mathbf{k}}k_xE_x)\\ &= \frac{\sigma}{en}(k_z^2T_en_{\mathbf{k}} - nek_z^2\varphi_{\mathbf{k}} - ik_zeE_zn_{\mathbf{k}})\end{aligned} \tag{IV. 161}$$

Expressing $\varphi_{\mathbf{k}}$ in terms of $n_{\mathbf{k}}$ and using the linearised ion eqn. (IV.160) it is straightforward to obtain the additional " frictional force "

$$F_y = \langle en'E_y' \rangle + \frac{eH}{c} \langle n'v_x' \rangle$$

and from this the additional transverse current $\delta j_x = \frac{c}{H} F_y$

$$\delta j_x = \sigma \int \frac{(k_x^2 k_z^2 E_x - k_x k_z k^2 E_z) \frac{\omega^2}{\Omega_i^2}}{(\Omega_e \tau_e)^2 k_z^4 + \omega^2 k^4/\Omega_i^2} \frac{N_{\mathbf{k}}}{n^2} \, d\mathbf{k} \qquad \text{(IV. 162)}$$

(we assume that the frequency ω is real).

For isotropic oscillations this gives $\delta j_x \sim \gamma \sigma E_x \left(\frac{\omega}{\Omega_i \Omega_e \tau_e} \right)^{\frac{3}{2}}$, which is considerably smaller than the value obtained above for high frequency oscillations. Even perturbations greatly extended along the magnetic field do not lead to a marked change in conductivity. At the same time oscillations of this type may lead to a considerable flux of charged particles

$$q_x = \langle v_x' n' \rangle \approx \frac{c}{H} \langle E_y' n' \rangle$$

that is to turbulent diffusion. The smallness of δj_x and F_y immediately shows that the diffusion flux is determined by the electric drift, i.e.

$$v_x' \approx \frac{cE_y'}{H}$$

From relation (IV.161), we can also determine the change in longitudinal current due to the inhomogeneous conductivity of the plasma; the result is

$$\delta j_z = -\sigma \int \frac{\Omega_e^2 \tau_e^2 k_z^4 E_z + \frac{\omega^2}{\Omega_i^2} k^2 k_x k_z E_x}{\Omega_e^2 \tau_e^2 k_z^4 + \frac{\omega^2}{\Omega_i^2} k^4} \frac{N_{\mathbf{k}}}{n^2} \, d\mathbf{k} \qquad \text{(IV. 163)}$$

which shows that the effect is again small.

5. Turbulent Plasma in Experimental Conditions

During plasma experiments one encounters turbulence effects at every step, and there is now available very extensive experimental data referring to situations in which turbulent motions occur to various degrees. Unfortunately, the turbulent motions themselves have not yet been investigated in any detail, the presence of a turbulence being deduced as a rule from its macroscopic manifestations. It is clear that the study of only macroscopic or averaged characteristics of the turbulent motion and not the spectrum and amplitude of the oscillations will not permit an unequivocal determination

of the type of turbulent motion which is responsible for any given macroscopic effect. At the present stage of research into the collective processes taking place in plasmas, it is not therefore possible to carry out a complete comparison between theory and experiment, and we must limit the discussion here to brief survey of the main experimental data.

(a) *Anomalous Diffusion*

The very first experiments with electric arcs in magnetic fields (5) showed that in a strong magnetic field fluctuations of fairly large amplitude are regularly excited in a plasma, and these considerably increase the effective diffusion coefficient. Bohm, in analysing the results of these experiments, concluded that the diffusion coefficient decreases only as H^{-1}, while according to classical theory it ought to decrease as H^{-2}. Simon (108, 109) showed, however, that these experiments admit of a very much simpler interpretation in terms of the so-called " short circuiting " of the electron and ion currents through the electrodes; the results do not then disagree with classical theory. However, this explanation did not eliminate the problem, since subsequent experiments have revealed enhanced diffusion in a great variety of conditions.

So far the greatest progress in the understanding of the origin and character of the anomalous diffusion of a plasma across a magnetic field has been made in the case of the positive column of a glow discharge. The first measurements of the characteristics of the positive column in a magnetic field, made by Rokhlin (112), Cummings and Tonks (113), Reikhrudel and Spivak (114), and Bickerton and von Engel (115), showed that for small magnetic fields the diffusion is classical. Nedospasov (116) arrived at the same conclusion by measuring the length of the cathode region of an arc discharge, and so also did Vasileva and Granovskiĭ (117), who measured the diffusion flux to the wall of the discharge tube directly. However, extending the experiments to high magnetic fields, Lehnert (31) observed unexpectedly that at a critical value of the magnetic field H_c the diffusion flux began to increase with the magnetic field, finally reaching a saturation level. This effect has been studied in some detail by Lehnert and Hoh (32) and later by other authors (34, 35, 118–123). It has been explained theoretically by Kadomtsev and Nedospasov (33) on the basis of the current–convective instability. As we showed above (Section I.2), this instability drives the discharge into a helical form for $H > H_c$. When the magnetic field is further increased the helical discharge in turn becomes unstable, and the motion of the plasma becomes turbulent, as we described in Section IV.4. The results of a calculation based on mixing-length theory are in good agreement with experimental data obtained from glow discharges at pressures of the order of $10^{-1}-1$ mm. Hg.

At smaller values of the neutral gas pressure, when the mean free path of the charged particles is of the order of the tube radius, the diffusion approximation is not convenient. According to present theoretical ideas, instabilities of the drift type must arise in this case. This conclusion is confirmed experimentally. Figure 27 shows, for instance, the dependence of the ion current I_w to a wall probe on the magnetic field in a mercury vapour

discharge at low pressure ($p = 3.6.10^{-4}$ mm. Hg), measured by Artsimovich, Nedospasov and Sobolev. The critical magnetic field at which the relationship $I_w(H)$ deviates from the H^{-2} law is in satisfactory agreement with the theoretical condition for the formation of the ion-sound instability (IV.88). The fact that the ion current in the turbulent discharge ($H \gg H_c$) is of the order of magnitude of the current in the absence of a magnetic field also agrees with the ion-sound mechanism of the build-up of oscillations.

The enhancement of the diffusion flux as the magnetic field increases above some critical value has been observed also in Penning-type discharges (discharges with oscillating electrons), where the longitudinal current is absent (126–128). In this case, together with low frequency oscillations,

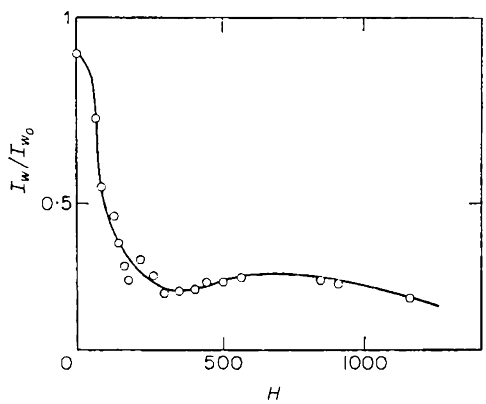

FIG. 27. Dependence of ion current to a wall probe on magnetic field in a mercury discharge at 3·6 . 10–⁴ mm. Hg pressure

noise at frequencies of the order of 10^9 cycles/sec is also observed, but this obviously is not directly related to the enhanced diffusion of ions. Chen and Cooper (128), using probes arranged along one magnetic line of force, have shown that the longitudinal phase velocity of the oscillations developing in such a turbulent plasma is of the order of 10^7 cm/sec, which considerably exceeds the velocity of sound $c_s \sim 10^6$ cm/sec. The frequency of the corresponding oscillations is of the order of 10^7 cycles/sec, and the relative amplitude of the density fluctuations is up to 50%. To judge by all these data, these oscillations can be considered as drift oscillations, and the corresponding instability as of the drift-dissipative type. Moreover, it has been observed by Simon (129) and Hoh (130) that in a Penning discharge there is an additional reason for an instability of the drift type related to the small difference between the drift velocities of the electrons and ions in a transverse electric field (an analogous effect in the case of a positive column has a stabilising influence (90)).

The experimental work of Geller (131), who studied the diffusion of a plasma in a high frequency discharge also belongs to this group of investigations. Geller also observed an increase of diffusion for $H > H_c$. In this case oscillations were excited in the radio frequency region. According to Geller,

the value of the critical field is inversely proportional to the tube radius a. Points corresponding to the observed values of the critical field are shown in Fig. 23, where they fall in the region of the ion-sound instability.

As we mentioned above, one of the earliest experiments on the diffusion of a plasma in a magnetic field was performed by Bohm, Burhop and others (5) in the study of ion sources. In these experiments the plasma was set up in a conducting anode chamber A (Fig. 28) by a beam of primary electrons with energy approximately 200 eV, which ionised the neutral gas at a pressure in the range 10^{-4}–10^{-2} mm. Hg. The plasma so formed diffused slowly across the magnetic field, at the same time spreading along the lines of force and recombining at the chamber ends. The transverse diffusion coefficient

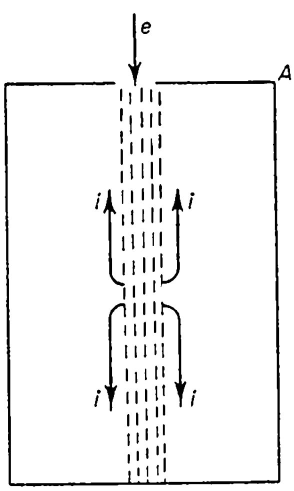

FIG. 28. Diffusion of ions in an arc discharge

may be determined from the characteristic length of the radial fall-off of the plasma density. Simon (103) showed that because of the " short circuiting " of the electron and ion currents by the conducting chamber, this length is determined by the ion diffusion coefficient which exceeds the electron diffusion coefficient by two orders of magnitude. When this effect is included, the experimental results can be explained on the basis of classical (laminar) concepts. The electron diffusion coefficient cannot be estimated from measurements of the radial density distribution alone. The higher electron current to a positively charged probe described in (5) does, however, support the existence of enhanced diffusion of the electrons, most probably related to oscillations with frequencies of 20–60 kc/s which are observed in these experiments. Subsequent experiments with arcs showed that the diffusion of the plasma in these conditions may in fact considerably exceed the classical value. For instance, Neidigh and Weaver (132) have found that under certain conditions the arc may change over into a regime referred to by these authors as " mode 2 " in which a rotating flare is ejected. They assumed that the appearance of the flare is related to variations of the neutral gas pressure

over the length of the discharge. However, experiments by Zharinov (133–135) have shown that the formation of one or several flares is also possible when the pressure is uniform. On the other hand, Boeschoten and Schwirzke (136) found that in similar conditions the diffusion was classical.

Experiments with arcs are difficult to discuss theoretically because of the processes occurring at the ends. Simon and Guest (137) showed that the longitudinal ion current to the ends in such a discharge may give rise to a helical current–convective instability. The critical magnetic field corresponding to the onset of this instability agrees satisfactorily with the experimental data. However, this is probably only one of many possible reasons for the instability of the plasma and perhaps far from the most important, although there is scarcely any reason to doubt that effects of this general type are in fact responsible for the instability; these effects were discussed particularly in Section IV.3.

We shall now turn to a group of experiments on the decay of a weakly ionised plasma. Excluding from the discussion the work of Bostick and Levine (29) referred to earlier, in which the toroidality of the magnetic field is important, we find that the papers (138–144) belong to this group. In these studies the plasma has a charged particle concentration in the range of 10^7–10^{10} cm^{-3}. If the magnetic field is not very large the measurements of the diffusion loss of particles agree with, or at least do not contradict, the classical diffusion theory. However, for high values of the magnetic field Golant and Zhilinskiĭ have observed an enhanced diffusion. According to their results (111), at magnetic fields exceeding 300 Oersted the diffusion coefficient of the charged particles in helium can be approximated with an accuracy of 30–40% by the empirical formula

$$D_\perp = 6 + \frac{(0.4+p)}{H^2} 10^8 \tag{IV. 164}$$

One of the terms represents approximately the classical value $D_c \approx 10^8 p/H^2$. The first term, independent of H, may be related to volume elimination processes (e.g. volume recombination, electron capture by impurities with subsequent recombination, etc.). The second term represents some additional diffusion mechanism not directly related to electron–atom collisions and depending on the magnetic field. Since these experiments were carried out at magnetic fields and pressure close to the theoretically predicted region of drift-dissipative instability (Fig. 23), we might suppose that this anomaly is related to an instability of this type.

The next group of experimental papers are concerned with the diffusion of a fully ionised plasma. The most comprehensive data on diffusion have been obtained on the B-1 and B-3 Stellarators. Ellis *et al.* (145) found that the loss of a fully ionised plasma from the Stellarator exceeds the classical losses by three to four orders, so that it cannot be attributed to electron–ion collisions. The energising of additional windings, which should stabilise the hydromagnetic instability of an ideal (infinitely conducting) plasma, showed no effect whatever on the containment time of the plasma, suggesting the

existence of a previously unknown instability mechanism. More detailed investigations by Stodiek *et al.* (146) showed that both during the ionisation stage and during the decay stage of the plasma, the charged particle loss rate agrees with Bohm's diffusion coefficient, that is, the measured values of the diffusion coefficient could be described empirically by the expression

$$D_{\perp} \approx 2{\cdot}10^4 \frac{T_e}{H} \qquad \text{(IV. 165)}$$

where the electron temperature is expressed in eV and the magnetic field in kilogauss.

To verify whether the plasma losses could be ascribed to the ion-sound instability (76, 147), Motley (148) studied the decay of a cold plasma in Stellarator B-1. In the absence of a longitudinal current the decay of the plasma was determined by the volume recombination if the electrons and ions, and the decay time was approximately 2 msec. However, when an additional electric field (with frequency 20 kc/sec and amplitude 0·01–0·03 V/cm) was applied, setting up in the plasma an appreciable longitudinal current, the decay rate increased sharply. A critical current level was found, and the corresponding longitudinal drift velocity of the electrons turned out to be of the order of the sound velocity, i.e. the velocity of the ions at the electron temperature. For small values of the ion temperature this result, it would seem, corresponds to the ion-sound instability (75, 76). However, the same values of the critical velocity were measured even when the temperature of the electrons was of the order of the ion temperature, so that the ion sound waves could not propagate because of strong damping at the ions. These results agree qualitatively with the theoretical concept of the drift instability (Section IV.4). However, it would be premature to describe this as complete agreement between theory and experiment, since on the one hand the turbulent diffusion theory is still in its early stages, and on the other there is as yet no complete understanding of the effects of small departures from equilibrium of the plasma column (see Section 5(c) below).

The results obtained on the Stellarator, according to which the diffusion of a currentless plasma is comparable to the diffusion of a plasma at rest, are confirmed by other experiments with a totally ionised plasma. Thus, D'Angelo and Rynn (149, 152) showed that the diffusion coefficient of potassium and caesium plasmas in a magnetic field agrees with the classical value up to almost 10,000 gauss. Noting that the theoretical diffusion coefficient (IV.139) for this case exceeds the classical value by only one order of magnitude, and taking into account the approximations in the theoretical estimate, we might think that there is no contradiction between theory and experiment. It is interesting to note that in a caesium and potassium plasma device, spontaneously excited drift waves have been observed (151,152). The authors relate the excitation of the waves to the existence of ion sheaths at the ends of the device.

Golant and Zhilinskiĭ (171, 153) also found that the diffusion of a decaying plasma at charged particle concentrations of the order of 10^{10}–10^{12} cm^{-3},

where electron–ion collisions are dominant, agrees well with the classical coefficient of diffusion at magnetic fields up to 1500 gauss.

Although we have not been able to discuss individually every diffusion experiment performed so far, it is clear that the consensus of the results is that even in a homogeneous magnetic field and in the absence of external beams, the diffusion coefficient both in a weakly ionised and in a fully ionised plasma may considerably exceed the classical value obtained from binary collisions. In the presence of a longitudinal current the diffusion coefficient is considerably higher than in a currentless plasma. In the weakly ionised plasma of a glow discharge, it is of the order of magnitude of the diffusion coefficient without a magnetic field, and in a fully ionised plasma it approximates to Bohm's coefficient. In the absence of a longitudinal current, the coefficient of diffusion is considerably smaller: in a fully ionised plasma at moderate magnetic fields, it is of the order of the classical coefficient, but in a weakly ionised plasma it may still be considerably higher than the classical value.

All these results either agree satisfactorily with approximate theory given in Section IV.4 or at least do not strongly contradict it. In order to achieve complete agreement between theory and experiment, however, a wide range of investigations is still necessary, both theoretical, to calculate rather than only estimate the effective diffusion coefficient, and experimental, to reveal the particle loss mechanism in its pure form.

(b) *Turbulent Heating of a Plasma*

By turbulent heating of a plasma—a term having a distinctly applied shade—we shall understand the transfer of ordcred energy, either the energy of charged particle beams or the energy of discrete oscillations, into energy of random motion and ultimately into heat, due to turbulence, i.e. the non-linear interaction between oscillations. The possibility of turbulent heating of a plasma in this sense was predicted qualitatively by Buneman (154) who discussed the excitation of oscillations in the plasma due to a beam instability under conditions in which all the electrons move relative to the ions. Such a state can be achieved, for instance, by means of a strong external electric field leading to electron " runaway ". Buneman showed by numerical calculations that the development of oscillations is accompanied by the scattering of the electrons at the inhomogeneities of the electric potential and the transformation of the directed motion of the electrons into random motion.

The Maxwellisation of the electron distribution function due to a collective interaction has been studied numerically by Dawson (155) in a one-dimensional model in which a charged plane or surface corresponds to each charged particle. Figure 29 shows a picture of the Maxwellisation of two oppositely directed electron beams obtained by that author.

Shapiro (156, 157) studied the initial stages of the randomisation of the directed velocity of the electrons by means of moment equations obtained by integration of the kinetic equation multiplied by I, $\mathbf{v}$ and v^2, over the velocity.

He showed that when the electron density in the beam is considerably smaller than the plasma density, an initial relatively short stage in which the beam spreads in velocitv is followed immediately by a longer stage of retardation, to which the quasi-linear approximation is applicable. The interaction between the beam and the oscillations leads to the formation of a plateau in the distribution function. One-third of the energy of the beam, which is equally distributed between the kinetic and electrostatic energies of the oscillations, is spent in the formation of the plateau.

Experimental data on the interaction of beams with plasmas have accumulated over many years. Effects of the collective interaction of an electron beam with a plasma were observed by Langmuir (1). The related anomalously rapid Maxwellisation of the electrons has been given a special name of

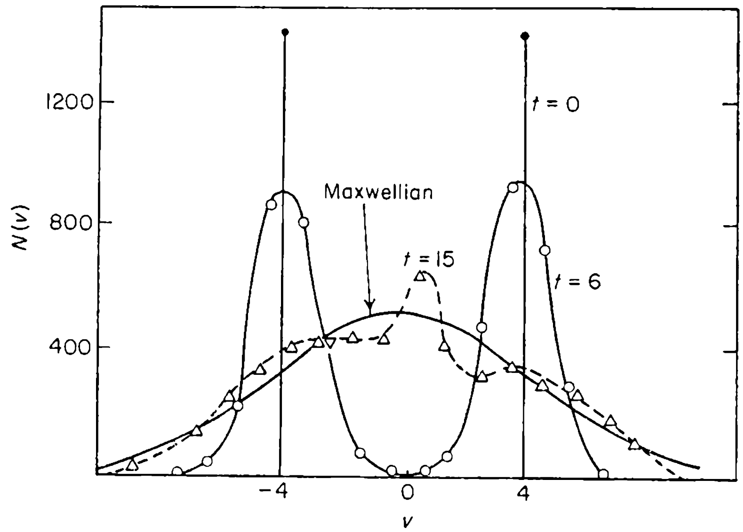

FIG. 29. Maxwellisation of two interpenetrating beams (a one-dimensional model)

" Langmuir's paradox ". The interaction of the beam with the plasma has been investigated in greater detail by Merrill and Webb (3). Using probe measurements they showed that during the passage of a beam localised regions of strong scattering are set up in the plasma. This effect was interpreted by Bohm and Gross (158) as a result of the modulation of the electron beam by the oscillating potential of a double layer formed at the plasma boundary. Then, as in the operation of a klystron, the initial perturbation of the density increases because the faster electrons overtake the slower ones, and at some point the electrostatic interaction between the electrons leads to their scattering. The scattered electrons which pass through the double layer at the boundary may slightly change its potential and give rise to feed-back between the oscillations at the entry point and in the region of strong scattering. Moreover, the oscillations of the double layer may lead to a transfer of energy to the plasma electrons, contributing to their Maxwellisation. Large amplitude oscillations of the double layer, with a frequency of the order of the plasma frequency, were observed experimentally by Gabor (159)

which would seem to support this mechanism for the excitation of the oscillations. However, von Gierke *et al.* (160), who repeated Gabor's experiment, observed no oscillations at all except some weak low frequency oscillations. This result perhaps indicates the possibility of a second mechanism for the interaction between the particles and the oscillations, namely a quasi-linear growth of the waves in space due to the instability, with the simultaneous formation of a plateau in the electron distribution function. The formation of such a plateau was demonstrated, for instance, in ref. (161). However, in the same paper arguments are also put forward in favour of the klystron mechanism of the development of oscillations. All this suggests that in fact both instability mechanisms may operate and the interaction between the electrons and the waves for small electron density in the beam is described satisfactorily by the quasi-linear approximation.

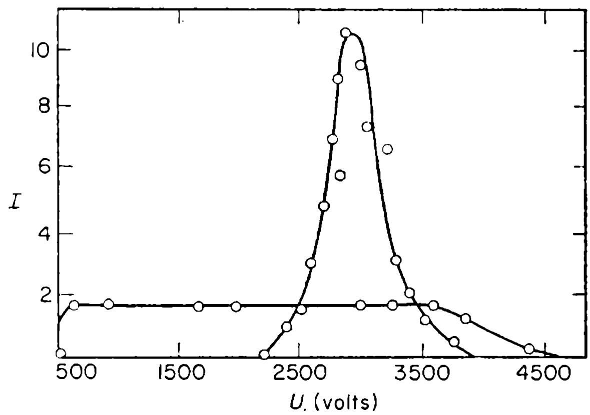

FIG. 30. Formation of plateau on distribution function

The quasi-linear effect of the formation of the plateau at the distribution function of the electrons is demonstrated in Fig. 30, which shows the electron distribution function before and after interaction with the plasma, measured experimentally in (162) (see also (163)). The heating of the plasma electrons during the interaction with the beam has been demonstrated in (165).

In practice, in the case of a bounded inhomogeneous plasma, the nature of the oscillations may be very much more complex than is usually assumed in theoretical discussions. In particular, when there are widely spaced resonance frequencies of the plasma system, strong excitation of separate harmonics may take place. Effects of this kind have been observed experimentally by Looney and Brown (4) in the investigation of high frequency plasma oscillations and by Alexeff and Neidigh (164) studying the excitation of ion-sound oscillations. Thus the real picture of the excitation of waves by powerful beams and the interaction of the waves with the particles and with one another may turn out to be much more complex, and to require a considerable further development of the theory for its complete description.

The excitation of ion sound by a current in the plasma also belongs to the group of phenomena considered in this section. Such oscillations have

been observed by Nedospasov (166) in a helium discharge at a fairly low neutral gas pressure. Both the order of magnitude of the steady state oscillations ($\sim 10^{-3}$ V) and the spectrum of the oscillations, in which the maximum occurred in the region of very low frequencies ($\omega \ll \Omega_0$), are in very good agreement with theoretical ideas (Section IV.2). The excitation of ion-sound oscillations has also been observed by Crawford (167) who studied the noise in a mercury discharge. Crawford observed that as the discharge current was increased, separate oscillations at discrete frequencies appear first, which then merge into a continuous spectrum in the range 10^4–10^6 cycles/sec. The phase velocity of these oscillations along the tube was 10^7–10^9 cm/sec. These are interpreted by the author as ion-sound waves propagating almost across the discharge. The excitation of such waves agrees with the theoretical picture given in Section IV.2. By using additional grids, Crawford showed that the source of the noise is the cathode (a similar effect is described in ref. (166)). However, the picture of the developed oscillations is probably not very sensitive to the form of the source, and therefore a comparison of these oscillations with a theoretical spectrum is justified.

The excitation of ion cyclotron oscillations by a longitudinal electric current, a physically similar effect, has been described in ref. (150). The excitation of ion oscillations obviously also explains the anomalous heating of the ions in Zeta (see following section). As we showed in Section IV.2, the ion heating may be related to an increased (" anomalous ") resistance of the plasma, arising from an additional frictional force between the electrons and ions due to the oscillations. An " anomalous " resistance has been observed on Zeta, and has also been demonstrated experimentally by Thomassen (169) in an experiment specially set up for this purpose. A similar effect has been observed by Adlam and Holmes (170) in the investigation of the dynamics of a super-fast pinch. These authors established that for a very fast rise of the current ($t \sim 10^{-8}$ sec) the thickness of the current skin is independent of time and about one-and-a-half orders of magnitude greater than the expected value c/ω_0. This result can be interpreted either as the result of a considerable decrease in the effective density of the electrons participating in the current (such a situation might arise from the " capture " of most of the electrons by the potential wells φ associated with the ions), or because of the suggested impossibility of increasing the directed velocity above some small fraction of the thermal velocity. The second possibility might arise from the excitation of the ion-sound instability. Adlam and Holmes support this point of view. However, the whole effect takes place during an extremely small interval of time which is obviously insufficient for the development of the ion-sound instability, which would indicate rather purely electronic oscillations which may be connected with ion perturbations existing in the plasma before the application of the longitudinal electric field pulse. In this case only the small proportion of the electrons " beyond the barrier " can carry the current. For a complete discussion of this extremely interesting and complex problem additional work will be required both in theoretical and experimental directions.

In an even clearer form, the " anomalous " resistance effect appears in the so-called " break " in the current during the last stage of a toroidal discharge (see the following section). The capture and acceleration of ions by powerful electron beams are associated with a similar effect, observed experimentally by Plutto (171). A complete theory of phenomena of this kind, most probably related to strong turbulence in the electron gas, has not yet been put forward.

We now go on to consider experiments in which turbulent processes were used for heating the plasma. One of these (165) has already been mentioned. The anomalous heating of the ions in Zeta (188) was a pleasant surprise. One might think that the cyclotron heating of the electrons described in reference (173) would also involve the excitation of a large number of degrees of freedom. Turbulent heating of the ions in a transverse electric field also occurred in Ioffe's experiments (174), where in addition to the heating due to the centrifugal instability of the rotating plasma, heating could have occurred due to instability of the plasma filament penetrated by the electron beam emitted from the cathode. The latter instability and the correponding ion heating has been investigated by Nezlin (175), (176) who observed that for a fairly high current density in the beam, when the directed flux amounts to approximately 1/6 of the random flux env_e, periodic cut-off of the beam occurs. The fluctuations of the electric field associated with this cutting-off of the beam lead to intense heating of the ions. This effect has not yet been interpreted theoretically. On the one hand, it might be considered as the result of the development of the two-stream ion-electron instability in some region where the instantaneous density was fairly small due to the ion oscillations (which may be excited by the Langmuir oscillations of the electron beam (177)). Yet on the other hand, the effect is very similar to the electrostatic cutting-off of a compensated electron beam (178), (179).

Turbulent heating due to oscillations of the magnetosonic type has been investigated by Zavoiskiĭ *et al.* (180–183). In these experiments, heating was achieved by superimposing on the main longitudinal magnetic field $H_z \sim 1000$ gauss a high frequency field with frequency $f \sim 10^7$ cycles and amplitude $\tilde{H} \sim 500$ gauss. As a result of the turbulent heating, the electron temperature increased to 500 eV and under certain conditions the ions were heated to a temperature of ~ 100 eV. The authors interpret the heating of the plasma as the result of the development of a two-stream electron-ion instability. The decay of the magnetosonic oscillations may also be important (see Section II. 1(b)).

The experimental results summarized above indicate that the use of collective processes for heating a plasma holds considerable promise.

(c) *Toroidal Discharges*

A whole series of turbulent processes has been observed during experiments with high temperature plasmas in toroidal devices. We have already noted above that anomalously fast plasma loss is observed in figure-8 shaped Stellarators, the so-called "pump out" effect. The detailed investigations of

this effect (146, 148) have shown that the anomalous diffusion is related to the longitudinal current, and the effective diffusion coefficient is of the order of the Bohm value. These results agree quantitatively with theoretical predictions based on the drift instability mechanism (see Section IV, 4). However, subsequent experiments made on Stellarator C (184–186) showed that the plasma loss may also be related to small departures from equilibrium of the plasma column due to inaccuracy of the magnetic field. For very small departures from the strict equilibrium conditions, the plasma column may be able to preserve an equilibrium by the closure of the confining currents through the diaphragm limiting the plasma column. The maximum possible value of the current that can be closed in this way is determined by the magnitude of the plasma loss. Under ideal equilibrium conditions this loss is symmetrical around the small circumference. When the equilibrium is upset, a predominantly ion current flows to one half of the diaphragm and an electron current to the other half, and the difference between these represents the transverse current $j_\perp$ necessary to maintain the equilibrium. For a sufficiently strong perturbation of the equilibrium condition (which would be represented, for example, by imposing a transverse field of only a few tenths of a per cent of the longitudinal field), the equilibrium loss current can no longer retain the plasma column which starts to move about over a large radius, which leads to additional plasma losses. Thus, the loss of particles and the deviation from the equilibrium position turn out to be inter-related. In the earlier experiments (146, 148) the equilibrium conditions were not controlled and some of these results may have to be revised considerably.

The effect of the transverse displacement of the plasma filament due to the perturbation of the equilibrium conditions was observed slightly earlier on a Tokomak device (186), where it was shown that to restore equilibrium additional current windings can be used setting up a transverse magnetic field. A similar application was also described in (184).

Another turbulent effect is the "cut-off" of the current of runaway electrons in an afterglow plasma, also observed on the Stellarator. Dreicer (187) has shown that this effect may be related to a build-up of Langmuir oscillations occurring due to the appearance of a minimum in the electron distribution function in the region between thermal and accelerated electrons. This minimum appears because of the faster retardation of slow electrons. However, this mechanism does not explain the appearance not of one, but of two or more steps as the current falls.

An unusual variety of different types of turbulent motion of a plasma are observed in discharges using moderate longitudinal currents. The most complete investigation of these processes has been made on Zeta (188–191). Amongst these belongs above all the anomalous heating of the ions. It was shown in ref. (189) that the mean energy of the impurity ions can be represented in the form $E_i = E_0 + \frac{m_i}{m_D} E_1$, where m_i is the mass of the ion, m_D the mass of the deuteron, and E_0 and E_1 are constants, $E_0 \sim 100$ eV, $E_1 \sim$ 2–3 eV. We can definitely conclude that the ions are in principle thermalised

and have a temperature of the order of 100 eV, which is several times higher than the electron temperature, and that they perform collective oscillations with a frequency considerably smaller than the cyclotron frequency, so that their total velocity $v_\perp = cE/H$ is independent of the mass, and the energy increases linearly with the mass. The heating of the ions may be due either to the current instability (for instance the excitation of magnetic sound oscillations considered in Section IV.2) or to a secondary effect occurring during turbulent convection, arising from the flute or dissipative instabilities. No detailed theoretical discussions of this problem is as yet available.

An effect closely related to the anomalous heating observed on Zeta is the excess resistance, which according to the experimental data increases linearly with the energy input to the plasma per particle (190). This additional resistance is of the same order as the ohmic resistance. Although we have noted effects of this type during our theoretical investigation (see Section IV. 2), a complete theory of the additional resistance has not yet been developed.

On Zeta and other devices using moderate magnetic fields, an anomalous particle loss is observed related to strong oscillations of the electric field (191). This loss may be related both to the convective (flute) instability and to the drift-dissipative instability. An estimate based on formula (IV.141) shows that in conditions in which perturbations greatly extended along the magnetic lines of force are admissible, the diffusion coefficient in the conditions obtaining in Zeta may exceed Bohm's value by two orders of magnitude. This value is quite sufficient to account for the observed losses. However, as the longitudinal wavelength is reduced the coefficient of diffusion according to (IV.143) decreases fairly rapidly. Therefore, in the outer regions of the discharge, where the effective length of a perturbation along the lines of force, being of the order of the pitch of the lines of force, is comparatively small, one would expect the primary instability to be the convective instability investigated in linear approximation by Suydam (192) It is possible that these loss mechanisms come into play and account for the " magnetic number " effect observed first on Zeta (190) and then on a smaller device (193). This effect consists in a prolonged conservation of a constant value of the pitch of the lines of force at the periphery of the discharge, the value being such that a line of force joins up to itself after one circuit of the torus, and abrupt transitions of the pitch from one such value to another as the discharge current is increased. One natural explanation of this effect put forward by Furth is to assume that on a magnetic surface with closed lines of force, where the longitudinal wavelength of the perturbation cannot be very large, the principal part in the transport of plasma is played by convective perturbations. During the rearrangement of the lines of force in such a perturbation the pitch is equalised over the radius, so that a region of constant pitch is generated, ultimately encompassing the entire periphery of the discharge. The enhanced diffusion of the aximuthal magnetic field into the discharge during its initial stages (194) also supports the existence of such a convective motion.

The anomalously rapid diffusion of a plasma in a discharge with moderate magnetic field leads to the establishment of a force-free configuration (195, 196). This configuration, which is not quite stationary but includes a small amplitude random motion due to the helical instability, can be described satisfactorily by an approximation in which it is assumed that because of the transverse convective motion of the plasma the apparent transverse conductivity vanishes (197). This approximation does not, however, account for the very interesting effect of the generation of reverse longitudinal magnetic flux at the outside of the plasma filament (195, 196, 198). This effect is hardly related to the toroidality, since in somewhat different conditions it is also observed on a straight discharge (201). The generation of this reverse field is probably related to the finite mixing length during the turbulent convection of the plasma (197). However, a quantitative theory of this effect has not yet been developed.

Another turbulent effect also observed first on Zeta (190), and later on a device of more modest size (193) is of exceptional interest. We have in mind the step-wise decline of the current during the last stage of the discharge. As on the Stellarator, there may be several steps in the current, but in contrast to the Stellarator, in this case the decline takes place in the presence of a longitudinal electric field which might sustain the discharge current. It might seem that in this case also this decline might be connected with the excitation of high frequency oscillations due to runaway electrons, the relative number of which increases considerably towards the end of the discharge. (Such oscillations have been observed experimentally on a straight discharge (200).) However, the fact that during the stepped decline of the current the magnetic energy is completely transformed into kinetic energy of the electrons, approximately 1 keV of energy being transferred to each electron, would rather seem to indicate an electron-ion collective process of the type of an anomalous resistance. A theory of this effect is also still absent.

We must also include in this section the work on the Levitron (199), a toroidal device with moderate longitudinal magnetic field and an additional current ring which should stabilise the hydrodynamic instabilities of the plasma. Oscillations and anomalous plasma loss are, however, still observed. These oscillations are obviously of a dissipative nature, related to the finite conductivity of the plasma.

(d) *Magnetic Traps*

The turbulent diffusion of a plasma in a trap with magnetic mirrors has been observed and studied in detail by Ioffe and others (174, 202–205). In these experiments a hot plasma with an ion energy of the order of 1 keV and particle density of the order of $10^9/\mathrm{cm}^{-3}$ was set up by accelerating ions in a pulsed electric field which was applied radially between the chamber walls and a cold plasma filament situated along the axis of the chamber. After the application of a high voltage pulse, the trap fills with plasma during 10–20 μsec. No detailed investigation of the motion of the plasma during this heating stage has yet been carried out, but we may suppose that the filling of the

chamber with plasma and the heating of the ions take place essentially due to the centrifugal instability of the rotating plasma. After the high voltage is switched off the rotation of the plasma stops and a more quiescent stage of turbulent convection due to the flute instability sets in. The presence of such a convection is evident chiefly from the anomalously rapid loss of plasma from the trap during a period of the order of 10^{-4} secs. Probe measurements have shown that the plasma decays essentially at the side walls of the trap, whole tubes of plasma lying along the magnetic field being ejected simultaneously towards the side walls. This result indicates that the fundamental mechanism of the convection is the flute instability. In a paper by this author (16) a semi-quantitative theory of such a convection has been developed, based on the mixing length concept. Further investigations of the relationship between the characteristic scale and amplitude of the turbulent pulsations and the distance to the wall, and the dependence of plasma lifetime on its density (203), supply additional confirmation of the theoretical model. Final proof of the flute instability mechanism was obtained when by the use of additional current conductors arranged at the periphery of the trap it was possible to stabilise the plasma and achieve prolonged containment (205).

It is interesting to note that the results on the turbulent convection were adequately described on the basis of hydrodynamic concepts without any consideration of the effect of the finite Larmor radius of the ions, i.e. of a collisionless viscosity, even though in the conditions of the experiments described in refs. (202, 203) this effect should be important. It is possible that it is smeared out by the strong inhomogeneity of the plasma brought about by the turbulent heating mechanism (see Section IV.4(j)), but the possibility cannot be excluded that the special features of the method used for producing the plasma in fact determine the entire subsequent motion. In fact, in the experiments of Ioffe *et al.*, considerable oscillations of the electric field take place during the phase in which the plasma is set up, which lead to the loss through the end of ions which are mirrored very near the surface of maximum magnetic field H_m. As a result, the plasma boundary moves towards the centre of the trap to a point where the magnetic field $H = H_s < H_m$. During small oscillations of the potential ϕ of the plasma tubes such that $\varphi < \frac{T_i}{e}\left(\frac{H_m}{H_s} - 1\right)$ all the ions continue to be contained by the mirrors, so that nothing prevents the development of the flute instability. With other more quiescent methods of heating the plasma, we might reach a situation in which the plasma fills the entire trap up to the centre of the mirrors where the magnetic field attains its maximum value $H = H_m$. In this case even small oscillations of the potential would lead to considerable currents to the ends due to the ejection of particles normally mirrored in the region of maximum magnetic field. These currents tend to stabilise the plasma, and this stabilisation effect is enhanced when there is a cold plasma surrounding the hot contained plasma. Most probably it is this effect which accounts for the stable containment of the plasma in adiabatic compression experiments in traps with magnetic fields (206, 207).

Amongst other collective effects observed in magnetic mirror traps, we may include the excitation of oscillations at very low density, where the transition to a quasi-neutral plasma begins (208, 209), and the excitation of ion cyclotron oscillations due to the anisotropy of their distribution function (210, 211). No detailed theory of these effects including the non-linear terms has been developed so far.

CONCLUSION

Summing up, we can state that during the past few years considerable progress has been made in developing a theory of plasma turbulence in the broad sense of this term, and in the understanding of the nature of the collective processes taking place in a plasma in laboratory conditions. A considerable number of problems concerning the development and interaction of oscillations in a plasma has been considered theoretically. Together with the study of weak turbulence, which is described satisfactorily by the kinetic wave equation, methods are being developed for the description of a strong turbulence including particularly the weak coupling method. A number of specific problems can be considered on a semi-empirical basis by the phenomenological introduction of the mixing length concept. At the present time, however, important fundamental difficulties are in sight which might delay the further development of the theory of plasma turbulence.

The situation is less satisfactory as far as the explanation of the experimentally-observed turbulent effects is concerned. Only for a small number of experiments is it possible to develop the corresponding theory and to obtain a satisfactory description. Yet in the majority of cases, not only has no quantitative theory of the observed effects been produced, but often even a qualitative interpretation of the nature of the observed effects is missing. This state of the matter is due partly to the rather indeterminate nature of the experimental data, but also to a considerable degree to the inadequate level of development of theoretical concepts and quantitative methods of describing collective processes. One must hope that it will be possible in the course of the next few years of hectic development of the theory, in conjunction with detailed and accurate experiments, to set up a complete picture of the turbulence of plasmas.

REFERENCES

1. Langmuir, I. *Phys. Rev.* **26** (1925), 585; *Z. Phys.* **46** (1927), 271.
2. Kaptsov, N. A. " Electrical Phenomena in Gases and in Vacua," (Elektricheskiye Yavleniya v Gazakh i Vakuume) (1947), Gostekhizdat, p. 490.
3. Merrill, H. J., and Webb, H. W. *Phys. Rev.* **55** (1939), 1191.
4. Looney, D. H., and Brown, S. C. *Phys. Rev.* **93** (1954), 965.
5. Bohm, D., Burhop, E., Massey, H. S. W., and Williams, R. " The Characteristics of Electrical Discharges in Magnetic Fields," (A. Guthrie and R. K. Wakerling, eds.) New York, 1949.
6. Sturrock, P. A. *Proc. Roy. Soc.* (*London*), **A242** (1957), 277.
7. Romanov, U. A., and Filippov, G. *Zhur. Eksptl i Teoret. Fiz.* **40** (1961), 123.
8. Vedenov, A. A., Velikhov, E. P., and Sagdeev, R. Z. *Nucl. Fusion*, **1** (1961), 82; *Suppl.* 1962, **72**, 465.
9. Drummond, W. E., and Pines, D. *Nucl. Fusion*, *Suppl.* 1962, Part 3, 1049.
10. Camac, M., Kantrowitz, A. R., Litvak, M. M., Patrick, R. M., and Petschek, H. F. *Nucl. Fusion*, *Suppl.* 1962, Part 2, 423.
11. Peierls, R. E., " Quantum Theory of Solids," Oxford, 1955.
12. Stix, T. H., " The Theory of Plasma Waves," McGraw-Hill, 1963.
13. Galeev, A. A., and Karpman, V. I. *Zhur. Eksptl i Teoret. Fiz.* **44** (1963), 592.
14. Kadomtsev, B. B., and Petviashvili, V. I. *Zhur. Eksptl i Teoret. Fiz.* **43** (1962), 2234.
15. Karpman, V. I. *Doklady Akad. Nauk, S.S.S.R.* **152** (1963), 587.
16. Kadomtsev, B. B. *Zhur. Eksptl i Teoret. Fiz.* **40** (1961), 328.
17. Kadomtsev, B. B. *Zhur. Tekh. Fiz.* **31** (1961), 1273.
18. Kraichnan, R. H. *J. Fluid Mech.*, **5** (1959) 497.
19. Tserkovnikov, U. A., *Zhur. Eksptl i Teoret. Fiz.* **32** (1957), 67.
20. Rudakov, L. I., and Sagdeev, R. Z., *Doklady Akad. Nauk, S.S.S.R.* **138** (1961), 581.
21. Rosenbluth, M. N., Rostoker, N., and Krall, N. A. *Nucl. Fusion*, *Suppl.* 1962, Part 1, 143.
22. Mikhailovskiĭ, A. B., Coll. " Problems in Plasma Theory ", (M. A. Leontovich, ed.) Vol. 3, 1963, Atomizdat.
23. Timofeev, A. V. *Zhur. Tekh. Fiz.* **33** (1963), 909.
24. Moiseev, S. S., and Sagdeev, R. Z., *Zhur. Eksptl i Teoret. Fiz.* **44** (1963), 763.
25. Landau, L. D., and Lifshits, E. M., " Mechanics of Continuous Media,," 1953 Gostekhizdat, M.
26. Sorokin, V. S., *Prikl.* **18** (1954), No. 1.
27. Stuart, I. I. *J. Fluid Mech.* **5** (1959), 209.
28. Timofeev, A. V. *Zhur. Tekh. Fiz.* **33** (1963), 776.
29. Bostick, W. H., and Levine, M. A. *Phys. Rev*, **97** (1955), 13.
30. Golant, V. E., Danilov, O. B., Zhilinski, A. P. *Zhur. Tekh. Fiz.* **33** (1963), 1043.
31. Lehnert, B. *in* " Proceedings of the Second Geneva Conference on the Peaceful Uses of Atomic Energy ", 1958, **32**, 349.

32. Hoh, F. C., and Lehnert, B. *Phys. Fluids*, **3** (1960), 600.
33. Kadomtsev, B. B., and Nedospasov, A. V. *J. Nucl. Energy*, Part C, **1** (1960), 230.
34. Allen, T. K., Paulikas, G. A., Pyle, R. V. *Phys. Rev. Letters*, **5**, (1960), 409.
35. Paulikas, G. A., Pyle, R. V. *Phys. Fluids*, **5** (1962), 348.
36. Lovetskiĭ, E. E., Kovrizhnikh, L. M., Rukhadze, A. A., Silin, V. P. *Doklady Akad, Nauk. S.S.S.R.* **149** (1963), 1052.
36a. Weinberg, S. *Phys. Rev.* **126**, (1962), 1899.
37. Sturrock, P. A. *Phys. Rev.* **112** (1959), 1488.
38. Sturrock, P. A. *Phys. Rev.* **117** (1960), 1460.
39. Fainberg, Ya. B., Kurilko, V. I., Shapiro, V. D. *Zhur. Eksptl i Teoret. Fiz.* **31** (1961), No. 6.
40. Polovin, R. V. *Zhur. Tekh. Fiz.* **31** (1961), 1220.
41. Feix, M. *Nuovo Cimento*, **27** (1963), 1130.
42. Taylor, J. B. *Phys. Rev. Letters*, **6** (1961) 262; *Nuclear Fusion Suppl.* Part 2 (1962), 477.
43. Vedenov, A. A. *Atomn. Energ.* **13** (1962) 5.
44. Akhiezer, A., Aleksin, V., Baryakhtov, V., and Peletminskiĭ, S. *Zhur. Eksptl i Teoret. Fiz.* **42** (1962), 552.
45. Akhiezer, A., Baryakhtov, V., and Peletminskiĭ, S. *Zhur. Eksptl i Teoret. Fiz.* **43** (1962), 1743.
46. Petviashvili, V. I. *Zhur. Eksptl i Teoret. Fiz.* **44** (1963), 1993.
47. Oraevskiĭ, V. N., and Sagdeev, R. Z. *Zhur. Tekh. Fiz.* **32** (1962), 1291.
48. Oraevskiĭ, V. N. *Zhur. Tekh. Fiz.* **33** (1963), 251.
49. Galeev, A. A., Oraevskiĭ, V. N. *Doklady Akad. Nauk, S.S.S.R.* **147** (1962), 71.
50. Balescu, S. *Phys. Fluids*, **3** (1960), 52.
51. Lenard, A. *Ann. Phys.* **3** (1960), 90.
52. Davidov, B. I. Coll. " Plasma Physics and Problems in Controlled Thermonuclear Reactions ", (Fizika Plazmy i Problema Upravlyaemykh Termoyadernykh Reaktsii) Vol. 1, p. 77, Moscow, 1958.
53. Klimontovich, Yu. L., and Silin, V. P. *Zhur. Eksptl i Teoret. Fiz.* **42** (1962), 986.
54. Silin, V. P., Gorbunov, L. M. *Doklady Akad. Nauk, S.S.S.R.* **145** (1962), 1265.
55. Silin, V. P. *Zhur. Eksptl i Teoret. Fiz.* **40** (1961), 1768.
56. Wild, H. W. *Ann. Phys.* **14** (1961), 143.
57. Batchelor, G. K. *J. Fluid. Mech.* **5** (1959), No. 1.
58. Novikov, E. A. *Doklady Akad. Nauk, S.S.S.R.* **139** (1961), 331.
59. Prandtl, L. *Z. angew. Math. Mech.* **5** (1925), 136; **22** (1942), No. 5.
60. Tollmien, W. *Z. angew. Math. Mech.* **6** (1926), 468.
61. Abramovich, G. N. " Theory of Turbulent Jets " (Teoriya Turbulentnikh Struĭ), Fizmatgiz, Moscow, 1960.
62. " Modern Developments in Fluid Dynamics," (S. Goldstein, ed.), 1943.
63. Kolmogorov, A. N. *Doklady Akad. Nauk, S.S.S.R.* **30** (1941), No. 4.
64. Obukhov, A. M. *Izvest. Akad. Nauk, S.S.S.R. Ser. Geogr. i Geofiz.*, No. 5 (1941).
65. Heisenberg, W. *Z. Physik.* **124** (1948), 628.
66. Thomson, W. B., and Hubbard, I. *Rev. Mod. Phys.* **32** (1960), 714.
67. Rostoker, N. *Nucl. Fusion*, **1** (1961), 101.
68. Hubbard, I. *Proc. Roy. Soc.* (*London*) **A260** (1960), 114; **A261** (1961), 369.
69. Wild, H. W., and Pines, D. *Phys. Rev.* **127** (1962), 1851.
70. Phillips, O. M. *J. Fluid Mech.* **4** (1958) 426.
71. Phillips, O. M., and Katz, E. J. *J. Mar. Res.* **19** (1961), No. 2.

72. Petviashvili, V. I. *Doklady Akad. Nauk, S.S.S.R.* **153** (1963), 1295.
73. Vedenov, A. A. "Problems in Plasma Theory" (Voprosy Teorii Plazmy) (M. A. Leontovich, ed.) No. 3 (1963), 203.
74. Silin, V. P. *Zhur. Eksptl i Teoret. Fiz.* **45** (1963), 816.
75. Gordeev, G. V. *Zhur. Eksptl i Teoret. Fiz.* **27** (1954) 19, 24.
76. Bernstein, I. B., Frieman, E. A., Kulsrud, R. M., and Rosenbluth, M. N. *Phys. Fluids,* **3** (1960), 136.
77. Jackson, E. A. *Phys. Fluids,* **3** (1960), 786.
78. Drummond, W. E., and Rosenbluth, M. N. *Phys. Fluids,* **5,** (1962), 1507.
79. Petviashvili, V. I. *Zhur. Eksptl i Teoret. Fiz.* **45** (1963), 1467.
80. Karpman, V. I. *Prikl. Mekh. Tekh. Fiz.* No. 6 (1963), 34.
81. Braginskiĭ, S. I., and Kazantsev, A. P., Coll. " Plasma Physics and Problems in Controlled Thermonuclear Reactions " (Fizika Plazmy i Problema Upravlyaemykh Termoyadernykh Reaktsii), Vol. 4, p. 24, 1958.
82. Stepanov, K. N. *Zhur. Eksptl i Teoret. Fiz.* **34** (1958), 1292.
83. Kadomtsev, B. B., and Timofeev, A. V. *Doklady Akad. Nauk, S.S.S.R.* **146** (1962), 581.
84. Mikhailovskiĭ, A. B., and Rudakov, L. I. *Zhur. Eksptl i Teoret. Fiz.* **44** (1963), 912.
85. Galeev, A. A., Oraevskiĭ, V. N., and Sagdeev, R. Z. *Zhur. Eksptl i Teoret. Fiz.* **44** (1963), 903.
86. Mikhailovskaya, L. V., and Mikhailovskiĭ, A. B. *Zhur. Tekh. Fiz.* **33** (1963), 1200.
87. Mikhailovskiĭ, A. B. *Nucl. Fusion,* **4,** No. 2 (1964).
88. Galeev, A. A. *Zhur. Eksptl i Teoret. Fiz.* **44** (1963), 1920.
89. Mikhailovskaya, L. V., and Mikhailovskiĭ, A. B. *Nucl. Fusion,* **3,** No. 1–2 (1963) 28, 113.
90. Timofeev, A. V. *Doklady Akad. Nauk, S.S.S.R.* **152** (1963), 84.
91. Kadomtsev, B. B. *J. Nucl. Energy,* Part C, **5** (1963), 31.
92. Zaslavskiĭ, G. M., Moiseev, S. S., and Oraevskiĭ, V. N. *Prikl. Mat. i Tekh. Fiz.* No. 6 (1963), 29.
93. Mikhailovskaya, L. V., and Mikhailovskiĭ, A. B. *Zhur. Eksptl i Teoret. Fiz.* **45** (1963), 1566.
94. Spitzer, L. *Phys. Fluids,* **1** (1958), 253.
95. Mikhailovskiĭ, A. B. *Zhur. Eksptl i Teoret. Fiz.* **43** (1962), 509.
96. Mikhailovskiĭ, A. B. *Zhur. Eksptl i Teoret. Fiz.* **44** (1963), 1552.
97. Mikhailovskiĭ, A. B., and Timofeev, A. V. *Zhur. Eksptl i Teoret. Fiz.* **44** (1963) 919.
98. Rosenbluth, M. N., and Krall, N. A. *Phys. Fluids,* **5** (1962), 1435.
99. Braginskiĭ, S. I., "Problems in Plasma Theory" (Voprosy Teorii Plazmy) (M. A. Leontovich, ed.) No. 1, p. 183, 1963, Atomizdat.
100. Rudakov, L. I. *Nucl. Fusion,* **2** (1962), 107.
101. Roberts, K. V., and Taylor, J. B. *Phys. Rev. Letters,* **8,** (1962) 197.
102. Galeev, A. A., Moiseev, S. S., and Sagdeev, R. Z. *Atomn. Energ.* **15** (1963), 451.
103. Kadomtsev, B. B. *Zhur. Eksptl i Teoret. Fiz.* **43** (1962), 1688.
104. Artsimovich, L. A., and Nedospasov, A. V. *Doklady Akad. Nauk, S.S.S.R.* **175** (1962), 1022.
105. Kadomtsev, B. B. *Zhur. Eksptl i Teoret. Fiz.* **45** (1963), 1231.
106. Galeev, A. A., and Rudakov, L. I. *Zhur. Eksptl i Teoret. Fiz.* **45** (1963), 647.
107. Yoshikawa, S., and Rose, D. J. *Phys. Fluids,* **5** (1962), 334.

108. Simon, A. *Phys. Rev.* **98** (1955), 317.
109. Simon, A. "Proceedings of the 2nd Geneva Conference on the Peaceful Uses of Atomic Energy" 1958, **31,** 315.
110. Hoh, F. C. *Rev. Mod. Phys.* **34** (1962), 267.
111. Golant, V. E. *Uspekhi. Fiz. Nauk*, **79** (1963), 377.
112. Rokhlin, G. N. *J. Phys.* U.S.S.R. **1** (1939), 347.
113. Cummings, C. S., and Tonks, L. *Phys. Rev.* **59** (1941), 517.
114. Reikhrudel, E., and Spivak, S. *J. Phys. U.S.S.R.* **4** (1941), 211.
115. Bickerton, R. J., and von Engel, A. *Proc. Phys. Soc.* (*London*), **69B** (1956), 468.
116. Nedospasov, A. V. *Zhur. Eksptl i Teoret. Fiz.* **34** (1958), 1338.
117. Vasileva, I. A., and Granovskiĭ, V. L. *Radiotekh. i Elektron.* **4** (1958), 2051.
118. Hoh, F. C. *Arkiv Fysik*, **18** (1960), 433.
119. Ekman, C., Hoh, F. C., and Lehnert, B. B. *Phys. Fluids*, **3** (1960), 833.
120. Zaitsev, A. A., and Vasileva, M. Ya. *Zhur. Eksptl i Teoret. Fiz.* **38** (1960), 1639.
121. Gierke, G. von, and Wohler, K. M. *Nucl. Fusion, Suppl.* 1962, Part I, 47.
122. Wohler, K. *Z. Naturforsch*, **17a** (1962), 937.
123. Vdovin, V. L., and Nedospasov, A. V. *Zhur. Tekh. Fiz.* **32** (1962), 817.
124. Bonnal, I. F., Briffod, G., and Manus, C. *Compt. rend.*, **250** (1960), 2859.
125. Bonnal, I. F., Briffod, G., and Manus, C. *Phys. Rev. Letters*, **6** (1961), 665.
126. Bonnal, I. F., Briffod, G., Manus, C., and Gregoire, M. *Nucl. Fusion Suppl.* 1962, Part 3, 995.
127. Reikhrudel, E. M., and Smirnitskaya, G. V. *Izv. VUZov, Radiofizika*, **1** (1958), 36.
128. Chen, F. F., and Cooper, A. W. *Phys. Rev.* **9** (1962), 333.
129. Simon, A. *Phys. Fluids*, **6** (1963), 382.
130. Hoh, F. C. *Phys. Fluids*, **6** (1963), 1184.
131. Geller, R., *Phys. Rev. Letters*, **9** (1962), 248.
132. Neidigh, R. V., and Weaver, C. H. "Proceedings of the 2nd Geneva Conference on the Peaceful Uses of Atomic Energy" 1958, **31,** 315.
133. Zharinov, A. V. *Atomn. Energ.* **7** (1959), 215.
134. Zharinov, A. V. *Atomn. Energ.* **10** (1962), 368.
135. Elizarov, L. I., and Zharinov, A. V. *Nucl. Fusion, Suppl.* 1962, Part 2, 699.
136. Boeschoten, F., and Schwirzke, F. *Nucl. Fusion*, **2** (1962), 54.
137. Guest, G., and Simon, A. *Phys. Fluids*, **5** (1962), 121.
138. Syrgiĭ, A. S., and Granovskiĭ, V. L. *Radiotekh. i Elektron.* **4** (1959), 1854.
139. Syrgiĭ, A. S., and Granovskiĭ, V. L. *Radiotekh. i Elekrton.* **5** (1960). 1522.
140. Golubev, V. S. *Radiotekh. i Elektron.* **7** (1962), 153.
141. Golant, V. E., and Zhilinskiĭ, A. P. *Zhur. Tekh. Fiz.* **30** (1960), 745.
142. Golant, V. E., and Zhilinskiĭ, A. P. *Zhur. Tekh. Fiz.* **32** (1962), 127.
143. Guzhova, S. K., and Syrgii, A. S. *Radiotekh. i Elektron.* **5** (1960), 1516.
144. Alikhanov, S. G., Zakharov, V. E., and Kharsanov, G. A. *Atomn. Energ.* **14** (1963), 137.
145. Ellis, R. A., Goldberg, L. P., and Gorman, J. G. *Phys. Fluids*, **3** (1960), 468.
146. Stodiek, W., Ellis, R. A., and Gorman, J. G. *Nucl. Fusion Suppl.* 1962, Part 1, 193.
147. Spitzer, L. *Phys. Fluids*, **3** (1960), 659.
148. Motley, K. W. *Nucl. Fusion, Suppl.* 1962, Part 1, 199.
149. D'Angelo, N., and Rynn, N. *Phys. Fluids*, **4** (1961), 275, 1303.
150. Motley, R. W., and D'Angelo, N. *Phys. Fluids*, **6** (1963), 296.
151. D'Angelo, N., and Motley, R. W. *Phys. Fluids*, **6** (1963), 422.
152. D'Angelo, N. *Nucl. Fusion*, **3** (1963), 147.

153. Golant, V. E., and Zhilinskiĭ, A. A. *Zhur. Tekh. Fiz.* **32** (1962), 1313.
154. Buneman, O. *Phys. Rev.* **115** (1959), 503.
155. Dawson, J. *Nucl. Fusion, Suppl.* 1962, Part **3**, 1033.
156. Shapiro, V. D. *Zhur. Tekh. Fiz.* **1** (1961), 522.
157. Shapiro, V. D. *Zhur. Eksptl i Teoret. Fiz.* **44** (1963), 613.
158. Bohm, D., and Gross, E. P. *Phys. Rev.* **70** (1949), 1801.
159. Gabor, D. Trans.IRE. **AP-4** (1956) 526.
160. Gierke, G. von, Ott, W., and Schwirzke, F. " Proceedings of the 5th International Conference on Ionization Phenomena in Gases," Munich, **2** (1961), 1412.
161. Gabovich, M. D. *Zhur. Eksptl i Teoret. Fiz.* **36** (1959), 1025.
162. Kharchenko, I. F., Fainberg, Ya. B., Nikolaev, R. M., Kornilov, E. A., Lutsenko, E. I., and Pedenko, N. S. *Nucl. Fusion, Suppl.* 1962, No. **3**, p. 1101.
163. Berezin, A. K., Berezin, G. P., Bolotin, A. I., and Fainberg, Ya. B. *Atomn. Energ.* **14** (1963), 249.
164. Alexeff, I., and Neidigh, R. V., " Proceedings of the 5th International Conference on Ionization Phenomena in Gases, Munich," **2** (1961) 1523; *Phys. Rev.* **129** (1963) 516.
165. Alexeff, I., Neidigh, R. V., Peed, W. F., Shipley, E. D., and Harris, E. G. *Phys. Rev. Letters*, **10** (1963) 273.
166. Nedospasov, A. V. *Nucl. Fusion, Suppl.* No. **3** (1962), p. 1228.
167. Crawford, F. W. " Proceedings of the 5th International Conference on Ionization Phenomena in Gases, Munich," **1** (1961) 593.
168. Consoli, T., Le Gardeur, R., and Slama, L. *Compt. Rend.* **253** (1961), 1923.
169. Thomassen, K. I. *Phys. Rev. Letters*, **10** (1963), 80.
170. Adlam, J. H., and Holmes, L. S. *Nucl. Fusion*, **3** (1963), 62.
171. Plutto, A. A. *Zhur. Eksptl i Teoret. Fiz.* **39** (1960), 1589.
172. Oraevskiĭ, V. I. *Prikl. Mat. i Tekh. Fiz.* No. 6 (1963) 34.
173. Ard, W. B., Becker, M. C., Dandl, R. A., Eason, H. O., England, A. C., and Kerr, R. J. *Phys. Rev. Letters* **10** (1963), 87.
174. Ioffe, M. S., Sobolev, R. I., Telkovskiĭ, V. G., and Yushmanov, E. E. *Zhur. Eksptl i Teoret. Fiz.* **40** (1961), 40.
175. Nezlin, M. V. *Zhur. Eksptl i Teoret. Fiz.* **41** (1961), 1015.
176. Nezlin, M. V., and Solntsev, A. M. *Zhur. Eksptl i Teoret. Fiz.* **45** (1963), 840.
177. Fedorchenko, V. D., Rutkevich, B. N., Muratov, V. I., and Chernyi, B. M. *Zhur. Tekh. Fiz.* **32** (1962), 958.
178. Pierce, J. R. *J. Appl. Phys.* **15** (1944), 721.
179. Volosov, V. I. *Zhur. Tekh. Fiz.* **32** (1962), 566.
180. Zavoiskiĭ, E. K. *Atomn. Energ.* **14**, (1963), 57.
181. Babykin, M. V., Zavoiskiĭ, E. K., Rudakov, L. I., and Skoryupin, V. P. *Nucl. Fusion, Suppl.* No. 3 (1962) 1073; *Zhur. Eksptl i Teoret. Fiz.* **43** (1962), 1976.
182. Babykin, M. V., Gavrin, P. P., Zavoiskiĭ, E. K., Rudakov, L. I., and Skoryupin, V. A. *Zhur. Eksptl i Teoret. Fiz.* **43** (1962), 411, 1547.
183. Babykin, M. V., Gavrin, A. A., Zavoiskii E. K., Rudakov, L. I., Skoryupin, V. A., and Sholin, G. V. *Zhur. Eksptl i Teoret. Fiz.* **45** (1963), 605.
184. Yoshikawa, S., Sinclair, R. M., Kessler, J. O., and Harries, W. *Phys. Fluids*, **6** (1963), 932, 937.
185. Yoshikawa, S., Harries, W. L., and Sinclair, R. M. *Phys. Fluids*, **6** (1963), 1506.
186. Artsimovich, L. A., and Kartashev, K. B. *Doklady Akad. Nauk, S.S.S.R.* **146** (1962), 1305.

187. Dreicer, H., and Mjolsness, R. *Phys. Fluids*, **5** (1962), 1531.
188. Butt, E. P., Carruthers, R., Mitchell, J. I. T., Thonemann, P. C., Bird, M., Blears, J., and Hartill, E. R., "Proceedings of the 2nd Geneva Conference on the Peaceful Uses of Atomic Energy " 1958, **32,** 42.
189. Jones, B. B., and Wilson, R. *Nucl. Fusion Suppl.* **3** (1962), 889.
190. Burton, W. M., Butt, E. P., Cole, H. C., Gibson, A., Mason, D. W., Pease, R. S., Whiteman, K., and Wilson, R. *Nucl. Fusion Suppl.* **3** (1962), 903.
191. Rusbridge, M. G., Lees, D. J., and Saunders, P. A. H., *ibid* 895.
192. Suydam, B., " Proceedings of the 2nd Geneva Conference on the Peaceful Uses of Atomic Energy " 1958, **31,** 157.
193. Babichev, A. P., Karchevskiĭ, A. I., and Muromkin, U. A. *Zhur. Tekh. Fiz.* **33** (1963), 1400.
194. Lees, D. J., and Rusbridge, M. G. *Phys. Fluids*, **4** (1961), 63.
195. Lees, D. J., and Rusbridge, M. G. " Proceedings of the 4th International Conference on Ionization Phenomena in Gases ", Uppsala, 1959, **2,** 954.
196. Babichev, A. P., Kartevskiĭ, A. I., Muromkin, U. A., and Sokhomskiĭ, V. V. *Nucl. Fusion Suppl.* **2,** (1962), 635; *Nucl. Fusion*, **2** (1962) 84.
197. Kadomtsev, B. B. *Nucl. Fusion Suppl.* **3,** (1962), 969.
198. Afrosimov, V. V. *et al. Zhur. Tekh. Fiz.* **30** (1960), 1383.
199. Birdsall, D. H., and Colgate, S. A. *et al. Nucl. Fusion Suppl.* **3** (1962), 955.
200. Supruneko, V. A. *et al. Atomn. Energ.* **14** (1963), 349.
201. Komelkov, V. S., Skvortsov, V., Tereschenko, V. N., and Zerevitinov, S. " Proceedings of the 5th International Conference on Ionization Phenomena in Gases ", Munich, 1961, **2,** 2190.
202. Ioffe, M. S., Sobolev, R. I., Telkovskiĭ, V. G., and Yushmanov, E. E. *Zhur. Eksptl i Teoret. Fiz.* **39** (1960), 1602.
203. Ioffe, M. S., and Yushmanov, E. E. *Nucl. Fusion Suppl.* **1** (1962), 177.
204. Gott, U. V., Ioffe, M. S., and Telkovskiĭ, V. G. *Nucl. Fusion Suppl.* **3** (1962), 1045.
205. Baiborodov, U. T., Ioffe, M. S., Petrov, V. M., and Sobolev, R. I. *Atomn. Energ.* **14** (1963), 443.
206. Post R. F. " Proceedings of the 2nd Geneva Conference on the Peaceful Uses of Atomic Energy " 1958, **32,** 245.
207. Post, R. F., Ellis, R. E., Ford, F. C., and Rosenbluth, M. N. *Phys. Rev. Letters*, **4** (1960), 166.
208. Brevnov, N. N., Romanovskiĭ, M. K., and Tomashchuk, U. F. *Nucl. Fusion Suppl.* **1,** (1962), 289.
209. Damm, C. C., Foote, J. H., Futch, A. N., and Post, R. F. *Phys. Rev. Letters,* **10** (1963), 323.
210. Pistunovich, V. I. *Atomn. Energ.* **14** (1963), 72.
211. Kucheriaev, Yu, A., and Panov, D. A. *J. Nucl. Energy*, Part C, **5** (1963), 145.
212. Moiseev, S. S., and Sagdeev, R. Z. *Zhur. Tekh. Fiz.* **34** (1964), 248.
213. Silin, V. P. *Prikl. Mat. i Tekh. Fiz.* No. 1 (1964), 33.
214. Klimontovich, Yu. L. *Zhur. Eksptl i Teoret. Fiz.* **36** (1959), 1405; *Doklady Akad. Nauk, S.S.S.R.* **144** (1962), 1022; *Prikl. Mat. i Tekh. Fiz.* No. 1 (1964), 14.
215. Pines, D., and Schrieffer, J. R., *Phys. Rev.* **125** (1962), 804.

INDEX

A

Acoustic waves, *see* Waves
Adiabatic compression, 137
Alfvén waves, *see* Waves
Anomalous diffusion, *see* Diffusion
Arcs in magnetic fields, 124, 126–127
  " mode 2," 126

B

Beam—plasma interaction, 130
  instability, *see* Instability
Binary collisions, 47, 58

C

Caesium plasma, 128
Cherenkov effect, 47
Collisions, binary, 47, 58
Compression, adiabatic, 138
Conductivity, electrical, of turbulent plasma, 119–123
  thermal, 79
Convection, 7–15
  in positive column, 11–15
Cyclotron radiation, 48

D

Damping, non-linear, 65–69, 76
Debye length or radius, 47, 65, 80
  number, 45
Decay instability, 38
Decaying plasma, 10, 11
Diffusion, anomalous, 1, 28–30, 48, 78, 124–129
  turbulent, 3, 106–119, 123
  in velocity space, 48, 58
  in wave number space, 57, 66
Diffusion coefficient, 29, 78, 96
  Bohm value, 3, 29, 108, 109, 113, 115, 116, 128, 129, 134, 135
  classical, 115, 127
  turbulent, 107–119
  in weak turbulence, 107
  for ion acoustic instability, 107
Diffusion coefficient,
  in rarefied currentless plasma, 108–110, 119
  in rarefied current carrying plasma, 110–113
  in very low density plasma, 114
  in dense plasma, 114–116
  effect of finite length, 109–110, 116
  measurements of, 126–129
  in stellarator, 127–128
  in Caesium plasma, 128
Dielectric permeability, 19, 21, 28, 42, 47, 65, 69
Discharge, glow, *see* Positive column
  H.F., 125–126
  P.I.G., 125
  super-fast pinch, 132
  toroidal, 133–136
Dispersion relations, 74, 76, 80, 83, 92, 94, 95, 97
  decay type, 34, 35, 119
  non-decay type, 34, 35, 62, 76
Double layer 130
Drift instability, *see* Instability
  Larmor, 79
  waves, 79–84

E

Electron Larmor radius, 98, 99
Energy exchange time, 41
Enhanced diffusion, *see* Diffusion
Excitation, hard, 5, 6
  soft, 5, 6

F

Flow, non-linear, 103–106
Flute instability, *see* Instability
Four-wave processes, 35, 68

G

Glow discharge, *see* Positive column
Green's function, 50, 57

H

Hall current, 120
Hard excitation, 5, 6
H.F. discharge, 125–126

I

Instability, absolute, 24
  beam plasma, 15, 16, 130
  centrifugal, of rotating plasma, 133, 137
  convective, 24
  current-convective, 12, 86, 99, 102–103, 104, 124, 127
  drift, 4, 78–90, 102, 125, 128, 134
  drift-dissipative, 4, 79, 96–102, 115, 125, 127, 135
  flute, 90–93, 105–106, 135, 137
  ion acoustic, 68, 94–95, 97–100, 107, 125, 128, 131–132
  ion cyclotron, 74, 95–96
  of inhomogeneous plasma, 78–103
  of weakly ionised plasma, 96–102
  two-stream (electron-ion), 133
Ion acoustic waves, *see* Waves
  cyclotron waves, *see* Waves
  Landau damping, 81, 85, 88, 89
  Larmor radius, 78, 81–92, 105–106, 119, 137
  sound waves, *see* Waves

J

Jet, turbulent, 59–61, 118

K

Kinetic wave equation, 2, 31–55, 42–44, 139
Kolmogorov-Obukhov spectrum, 3, 51, 61–62

L

Landau collision term, 47
  damping, inverse, 16
    ion, 81, 85, 88, 89
    linear, 41, 44, 84, 86
    non-linear, 41, 68, 76
Langmuir paradox, 1, 130
  waves, *see* Waves
Larmor drift, 79
  radius, electron, 98, 99
    ion, 78, 81-92, 105–106, 119, 137
Levitron, 136

M

Magnetic mirror trap, 3, 136–138
" Magnetic number " effect, 135
Magneto-sonic waves, *see* Waves
Maxwell stress tensor, 28, 29
Mixing length, 3, 59–61, 116–118, 124, 137, 139

N

Navier-Stokes equation, 51
Non-linear damping, 65–69, 76
  flow, 103–106
  Landau damping, 41, 68, 76

P

Penning (P.I.G.) discharge, 125
Permeability, dielectric, 19, 21, 28, 42, 47, 65, 69
Perturbation theory, 51
Plateau in distribution function, 17, 72, 75–77, 131
Positive column, 3, 11–15, 116–118, 124–125
Potassium plasma, 128
Pump-out, 134

Q

Quasi-linear approximation, 2, 5–30, 72, 75, 131

R

Random phase approximation, 33, 46
Resistance, anomalous, 132, 135, 136
Resistivity, anomalous, 48, 77
  see also Conductivity, electrical
Reynolds' stress, 59
Runaway, of electrons, 129, 134, 136

S

Shear, of magnetic field, 89–90
Shock, collisionless, 2
Soft excitation, 5, 6
Sound waves, *see* Waves
Spectrum of turbulence, for ion acoustic waves, 68–72
  for wind waves, 62–63
  Kolmogorov-Obukhov, 3, 51, 61–62
Stellarator, 127–128, 133–136
Strong turbulence, 2, 3, 49–63, 116

T

Thermal conductivity, 79
  fluctuations, 44–46
  noise level, 48
Three-wave processes, 34
Tokomak, 134
Turbulence, fluid, 51
  strong, 2, 3, 49–63, 116
  weak, 2, 7, 18, 31–48, 49, 139
Turbulent, jet, 59–61

V

Van de Pol method, 6
Viscosity, 51, 61, 79
  collisionless, 105, 137

W

W.K.B. approximation, 23, 27
Waves, Alfvén, 2, 39–41, 76, 83
    accelerated, 83
    decelerated, 83
  drift, 79–84, 125
  ion acoustic, 35–39, 47, 68–73, 78, 81, 100, 131, 132
    accelerated, 81
    decelerated, 81
Waves, Alfvén
  ion cyclotron, 73–75, 132, 138
  ion sound, *see* Waves, ion acoustic
Waves, Langmuir, 2, 3, 15–18, 26, 28, 35–39, 42–44, 47, 51, 65–68, 134
  magneto-sonic, 2, 39–41, 75–78, 133
  sound, *see* Waves, ion acoustic
Wave equation, kinetic, *see* Kinetic wave equation
" Wave number," 67–71
Wave packets, 54–57
Wave propagation, 2, 7, 18, 31–49
  in inhomogeneous plasma, 18–23
Wave-particle interaction, 46–48
  adiabatic, 25–30, 48
  resonance, 17, 25–30, 48
Wave-wave interaction, adiabatic, 54, 55
  resonance, 54, 55, 57
Weak coupling approximation, 3, 49–58, 116
Weak turbulence, 2, 7, 18, 31–48, 49, 139
Whistlers, 76
Wind waves, 62–63

Z

Zeta, 132–136